8.2: Electric Dipole Radiation

Consider again the problem that was discussed in electrostatics (Sec. 3.1), namely the field of a localized source with linear dimensions $a << r$ (see Fig. 1 again), but now with time-dependent charge and/or current distributions. Using the arguments of that discussion, in particular the condition expressed by Eq. (3.1), $r' << r$, we may apply the Taylor expansion (3.3), truncated to two leading terms,

$$f(\mathbf{R}) = f(\mathbf{r}) - \mathbf{r}' \cdot \nabla f(\mathbf{r}) + \ldots, \tag{8.18}$$

to the scalar function $f(\mathbf{R}) \equiv R$ (for which $\nabla f(\mathbf{r}) = \nabla R = \mathbf{n}$, where $\mathbf{n} \equiv \mathbf{r}/r$ is the unit vector directed toward the observation point – see Fig. 1) to approximate the distance R as

$$R \approx r - \mathbf{r}' \cdot \mathbf{n}. \tag{8.19}$$

In each of the retarded potential formulas (17), R participates in two places: in the denominator and in the source's time argument. If ρ and $\mathbf{j}$ change in time on the scale $\sim 1/\omega$, where ω is some characteristic frequency, then any change of the argument $(t - R/\nu)$ on that time scale, for example due to a change of R on the spatial scale $\sim \nu/\omega = 1/k$, may substantially change these functions. Thus, the expansion (19) may be applied to R in the argument $(t - R/\nu)$ only if $ka << 1$, i.e. if the system's size a is much smaller than the radiation wavelength $\lambda = 2\pi/k$. On the other hand, the function $1/R$ changes relatively slowly, and for it even the first term of the expansion (19) gives a good approximation as soon as $a << r$, R. In this approximation, Eq. (17a) yields

$$\phi(\mathbf{r}, t) \approx \frac{1}{4\pi\varepsilon r} \int \rho\left(\mathbf{r}', t - \frac{R}{\nu}\right) d^3r' \equiv \frac{1}{4\pi\varepsilon r} Q\left(t - \frac{R}{\nu}\right), \tag{8.20}$$

where $Q(t)$ is the net electric charge of the localized system. Due to the charge conservation, this charge cannot change with time, so that the approximation (20) describes just a static Coulomb field of our localized source, rather than a radiated wave.

Let us, however, apply a similar approximation to the vector potential (17b):

$$\mathbf{A}(\mathbf{r}, t) \approx \frac{\mu}{4\pi r} \int \mathbf{j}\left(\mathbf{r}', t - \frac{R}{\nu}\right) d^3r'. \tag{8.21}$$

According to Eq. (5.87), the right-hand side of this expression vanishes in statics, but in dynamics, this is no longer true. For example, if the current is due to a non-relativistic motion[6] of a system of point charges q_k, we can write

$$\int \mathbf{j}\left(\mathbf{r}', t\right) d^3r' = \sum_k q_k \dot{\mathbf{r}}_k(t) = \frac{d}{dt} \sum_k q_k \mathbf{r}_k(t) \equiv \dot{\mathbf{p}}(t), \tag{8.22}$$

where $\mathbf{p}(t)$ is the dipole moment of the localized system, defined by Eq. (3.6). Now, after the integration, we may keep only the first term of the approximation (19) in the argument $(t - R/\nu)$ as well, getting

$$\mathbf{A}(\mathbf{r}, t) \approx \frac{\mu}{4\pi r} \dot{\mathbf{p}}\left(t - \frac{r}{\nu}\right), \quad \text{for } a << R, \frac{1}{k}. \tag{8.23}$$

Let us analyze what exactly this result describes. The second of Eqs. (1) allows us to calculate the magnetic field by the spatial differentiation of $\mathbf{A}$. At large distances $r >> \lambda$ (i.e. in the so-called far-field zone), where Eq. (23) describes a virtually plane wave, the main contribution into this derivative is given by the dipole moment factor:

$$\text{Far-field wave} \qquad \mathbf{B}(\mathbf{r}, t) = \frac{\mu}{4\pi r} \nabla \times \dot{\mathbf{p}}\left(t - \frac{r}{\nu}\right) = -\frac{\mu}{4\pi r \nu} \mathbf{n} \times \ddot{\mathbf{p}}\left(t - \frac{r}{\nu}\right). \tag{8.24}$$

This expression means that the magnetic field, at the observation point, is perpendicular to the vectors $\mathbf{n}$ and (the retarded value of) $\ddot{\mathbf{p}}$, and its magnitude is

$$B = \frac{\mu}{4\pi r \nu} \ddot{p}\left(t - \frac{r}{\nu}\right) \sin\Theta, \quad \text{i.e. } H = \frac{1}{4\pi r \nu} \ddot{p}\left(t - \frac{r}{\nu}\right) \sin\Theta, \tag{8.25}$$

where Θ is the angle between those two vectors – see Fig. 2.[7]

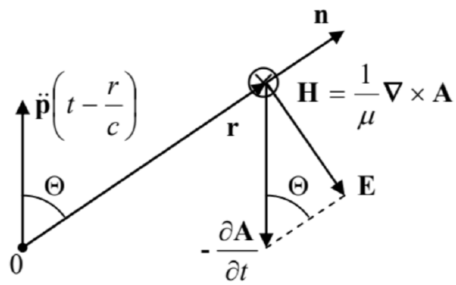


Fig. 8.2. Far-fields of a localized source, contributing to its electric dipole radiation.

The most important feature of this result is that the time-dependent field decreases very slowly (only as $1/r$) with the distance from the source, so that the radial component of the corresponding Poynting vector (7.9b),[8]

$$S_r = ZH^2 = \frac{Z}{(4\pi\nu r)^2}\left[\ddot{p}\left(t-\frac{r}{\nu}\right)\right]^2\sin^2\Theta, \qquad \text{Instant power density} \tag{8.26}$$

drops as $1/r^2$, i.e. the full instant power $\mathscr{P}$ of the emitted wave,

$$\mathscr{P} \equiv \oint_{4\pi} S_r r^2 d\Omega = \frac{Z}{(4\pi\nu)^2}\ddot{p}^2 2\pi\int_0^\pi \sin^3\Theta d\Theta = \frac{Z}{6\pi\nu^2}\ddot{p}^2. \qquad \text{Larmor formula} \tag{8.27}$$

does not depend on the distance from the source – as it should for radiation.[9]

This is the famous Larmor formula[10] for the electric dipole radiation; it is the dominating component of radiation by a localized system of charges – unless $\ddot{\mathbf{p}}=0$. Please notice its angular dependence: the radiation vanishes at the axis of the retarded vector $\ddot{\mathbf{p}}$ (where $\Theta=0$), and reaches its maximum in the plane perpendicular to that axis.

In order to find the average power, Eq. (27) has to be averaged over a sufficiently long time. In particular, if the source is monochromatic, $\mathbf{p}(t)=\mathrm{Re}[\mathbf{p}_\omega\exp\{-i\omega t\}]$, with a time-independent vector $\mathbf{p}_\omega$, such averaging may be carried out just over one period, giving an extra factor 2 in the denominator:

$$\overline{\mathscr{P}} = \frac{Z\omega^4}{12\pi\nu^2}|p_\omega|^2. \qquad \text{Average radiation power} \tag{8.28}$$

The easiest application of this formula is to a point charge oscillating, with frequency ω, along a straight line (which we may take for the z-axis), with amplitude a. In this case, $\mathbf{p}=qz(t)\mathbf{n}_z = qa\,\mathrm{Re}[\exp\{-i\omega t\}]\mathbf{n}_z$, and if the charge velocity amplitude, $a\omega$, is much less than the wave speed ν, we may use Eq. (28) with $p_\omega = qa$, giving

$$\overline{\mathscr{P}} = \frac{Zq^2a^2\omega^4}{12\pi\nu^2}. \tag{8.29}$$

Applied to an electron $\left(q=-e\approx-1.6\times10^{-19}\mathrm{C}\right)$, initially rotating about a nucleus at an atomic distance $a\sim10^{-10}$ m, the Larmor formula shows[11] that the energy loss due to the dipole radiation is so large that it would cause the electron to collapse on the atom's nucleus in just $\sim10^{-10}$ s. In the beginning of the 1900s, this classical result was one of the main arguments for the development of quantum mechanics, which prevents such collapse of electrons in their lowest-energy (ground) quantum state.

Another useful application of Eq. (28) is the radio wave radiation by a short, straight, symmetric antenna which is fed, for example, by a TEM transmission line such as a coaxial cable – see Fig. 3.

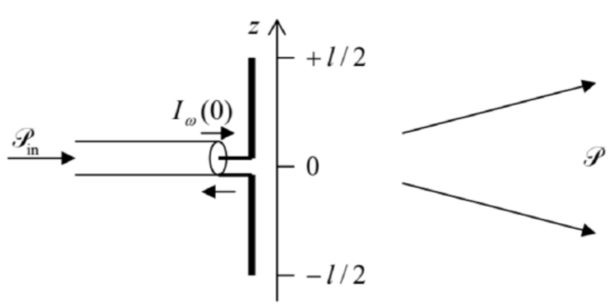


Fig. 8.3. The dipole antenna.

The exact solution of this problem is rather complicated, because the law $I_\omega(z)$ of the current variation along the antenna's length should be calculated self consistently with the distribution of the electromagnetic field induced by the current in the surrounding space. (This fact is unfortunately ignored in some textbooks.) However, one may argue that at $l << \lambda$, the current should be largest in the feeding point (in Fig. 3, taken for $z = 0$), vanish at antenna's ends $(z = \pm l/2)$, so that the linear function,

$$I_\omega(z) = I_\omega(0)\left(1 - \frac{2}{l}|z|\right), \tag{8.30}$$

should give a good approximation of the actual distribution – as it indeed does. Now we can use the continuity equation $\partial Q/\partial t = I$, i.e. $-i\omega Q_\omega = I_\omega$, to calculate the complex amplitude $Q\omega(z) = iI_\omega(z)\,\mathrm{sgn}(z)/\omega$ of the electric charge $Q(z,t) = \mathrm{Re}[Q_\omega \exp\{-i\omega t\}]$ of the wire's part beyond point z, and from it, the amplitude of the linear density of charge

$$\lambda_\omega(z) \equiv \frac{dQ_\omega(z)}{d|z|} = -i\frac{2I_\omega(0)}{\omega l}\mathrm{sgn}\, z. \tag{8.31}$$

From here, the dipole moment's amplitude is

$$p_\omega = 2\int_0^{l/2} \lambda_\omega(z) z dz = -i\frac{I_\omega(0)}{2\omega}l, \tag{8.32}$$

so that Eq. (28) yields

$$\overline{\mathscr{P}} = Z\frac{\omega^4}{12\pi\nu^2}\frac{|I_\omega(0)|^2}{4\omega^2}l^2 = \frac{Z(kl)^2}{24\pi}\frac{|I_\omega(0)|^2}{2}, \tag{8.33}$$

where $k = \omega/\nu$. The analogy between this result and the dissipation power, $\mathscr{P} = \mathrm{Re}\, Z|I_\omega|^2/2$, in a lumped linear circuit element, allows the interpretation of the first fraction in the last form of Eq. (33) as the real part of the antenna's impedance:

$$\mathrm{Re}\, Z_A = Z\frac{(kl)^2}{24\pi}, \tag{8.34}$$

as felt by the transmission line.

According to Eq. (7.118), the wave traveling along the line toward the antenna is fully radiated, i.e. not reflected back, only if Z_A equals to of the line. As we know from Sec. 7.5 (and the solution of the related problems), for typical TEM lines, $Z_W \sim Z_0$, while Eq. (34), which is only valid in the limit $kl << 1$, shows that for radiation into free space $(Z = Z_0)$, $\mathrm{Re}\, Z_A$ is much less than Z_0. Hence to reach the impedance matching condition $Z_W = Z_A$, the antenna's length should be increased – as a more involved theory shows, to $l \approx \lambda/2$. However, in many cases, practical considerations make short antennas necessary. The example most often met nowadays is the cell phone antennas, which use frequencies close to 1 or 2 GHz, with free-space wavelengths λ between 15 and 30 cm, i.e. much larger than the phone size.[12] The quadratic dependence of the antenna's efficiency on l, following from Eq. (34), explains why every millimeter counts in the design of such antennas, and why their designs are carefully optimized using software packages for the (virtually exact) numerical solution of the Maxwell equations for the specific shape of the antenna and other phone parts.[13]

To conclude this section, let me note that if the wave source is not monochromatic, so that $\mathbf{p}(t)$ should be represented as a Fourier series,

$$\mathbf{p}(t) = \mathrm{Re}\sum_\omega \mathbf{p}_\omega e^{-i\omega t}, \tag{8.35}$$

the terms corresponding to the interference of spectral components with different frequencies ω are averaged out at the time averaging of the Poynting vector, and the average radiated power is just a sum of contributions (28) from all substantial frequency components.

Reference

[6] For relativistic particles, moving with velocities of the order of speed of light, one has to be more careful. As the result, I will postpone the discussion of their radiation until Chapter 10, i.e. until after the detailed discussion of special relativity in Chapter 9.

[7] From the first of Eqs. (1), for the electric field, in the first approximation (23), we would get $-\partial \mathbf{A}/\partial t = -(1/4\pi\varepsilon\nu r)\ddot{\mathbf{p}}(t - r/\nu) = -(Z/4\pi r)\ddot{\mathbf{p}}(t - r/\nu)$. The transverse component of this vector (see Fig. 2) is the proper electric field $\mathbf{E} = Z\mathbf{H} \times \mathbf{n}$ of the radiated wave, while its longitudinal component is exactly compensated by $(-\nabla\phi)$ in the next term of the Taylor expansion of Eq. (17a) in small parameter $ka \sim a/\lambda << 1$.

[8] Note the "doughnut" dependence of S_r on the direction **n**, frequently used to visualize the dipole radiation.

[9] In the Gaussian units, for free space $(\nu = c)$, Eq. (27) reads $\mathscr{P} = \left(2/3c^3\right)\ddot{p}^2$.

[10] Named after Joseph Larmor, who was first to derive it (in 1897) for the particular case of a single point charge q moving with acceleration $\ddot{\mathbf{r}}$, when $\ddot{\mathbf{p}} = q\ddot{\mathbf{r}}$.

[11] Actually, the formula needs a numerical coefficient adjustment to account for the electron's orbital (rather than linear) motion – the task left for the reader's exercise. However, this adjustment does not affect the order-of-magnitude estimate given above.

[12] The situation will be partly remedied by the ongoing transfer of the wireless mobile technology to its next (5G) generation, with the frequencies moved up to the 28 GHz, 37-39 GHz, and possibly even the 64-71 GHz bands.

[13] A partial list of popular software packages of this kind includes both publicly available codes such as Nec2 (whose various versions are available online, e.g., at http://www.qsl.net/4nec2/), and proprietary packages – such as Momentum from Agilent Technologies (now owned by Hewlett-Packard), FEKO from EM Software & Systems, and XFdtd from Remcom.

LECTURE NOTES 13

ELECTROMAGNETIC RADIATION

In P436 Lect. Notes 4-10.5 (Griffiths *ch*. 9-10}, we discussed the ***propagation*** of macroscopic *EM* waves, but we have not yet discussed ***how*** macroscopic *EM* waves are ***created***. Using what we learned in P436 Lect. Notes 12, we can now discuss how macroscopic *EM* waves ***are*** created.

"Encrypted" into Maxwell's equations:

1) $\boxed{\vec{\nabla}\cdot\vec{E}(\vec{r},t) = -\frac{\rho_{tot}(\vec{r},t)}{\varepsilon_o}}$ 3) $\boxed{\vec{\nabla}\times\vec{E}(\vec{r},t) = -\frac{\partial\vec{B}(\vec{r},t)}{\partial t}}$

2) $\boxed{\vec{\nabla}\cdot\vec{B}(\vec{r},t) = 0}$ 4) $\boxed{\vec{\nabla}\times\vec{B}(\vec{r},t) = \mu_o\vec{J}_{tot}(\vec{r},t) + \mu_o\varepsilon_o\frac{\partial\vec{E}(\vec{r},t)}{\partial t}}$

is the physics associated with ***radiation*** of electromagnetic waves/electromagnetic energy, arising from the ***acceleration*** {and/or ***deceleration***} of electric charges (and/or electric currents).

In the P436 Lecture Notes #12, we derived the ***retarded*** electromagnetic fields associated with a moving point charge q from the ***retarded*** Liénard-Wiechert potentials:

$\boxed{V_{\rm r}(\vec{r},t) = \frac{q}{4\pi\varepsilon_o}\frac{1}{\kappa\mathscr{r}}}$ where: $\boxed{\mathscr{r} \equiv c\Delta t_r = c(t - t_{\rm r})}$, $\boxed{\vec{\mathscr{r}} \equiv \vec{r}(t) - \vec{r}'(t_{\rm r})}$

$\boxed{\vec{A}_{\rm r}(\vec{r},t) = \frac{\mu_o q}{4\pi}\frac{\vec{v}(t_{\rm r})}{\kappa\mathscr{r}}}$ and: $\boxed{\kappa \equiv 1 - \hat{\mathscr{r}}\cdot\vec{v}(t_{\rm r})/c = 1 - \hat{\mathscr{r}}\cdot\vec{\beta}(t_{\rm r})}$ = "retardation" factor

With: $\boxed{\vec{A}_{\rm r}(\vec{r},t) = \vec{\beta}(t_{\rm r})\left(V_{\rm r}(\vec{r},t)/c\right)}$ $\boxed{\vec{\beta}(t_{\rm r}) \equiv \vec{v}(t_{\rm r})/c}$ and: $\boxed{c^2 = 1/\varepsilon_o\mu_o}$

We also derived the corresponding ***retarded*** electric and magnetic fields associated with a ***moving point*** electric charge q:

$$\boxed{\vec{E}_{\rm r}(\vec{r},t) = -\vec{\nabla}V_{\rm r}(\vec{r},t) - \frac{\partial\vec{A}_{\rm r}(\vec{r},t)}{\partial t}}$$

$$\boxed{\vec{E}_{\rm r}(\vec{r},t) = \frac{q}{4\pi\varepsilon_o}\frac{\mathscr{r}}{\left(\vec{\mathscr{r}}\cdot\vec{u}(t_r)\right)^3}\left[\overbrace{\left(c^2 - v^2(t_{\rm r})\right)\vec{u}(t_{\rm r})}^{\text{term for generalized Coulomb field/velocity field}} + \overbrace{\vec{\mathscr{r}}\times\left(\vec{u}(t_{\rm r})\times\vec{a}(t_{\rm r})\right)}^{\text{term for radiation/acceleration field}}\right]}$$

$\boxed{\vec{B}_{\rm r}(\vec{r},t) = \vec{\nabla}\times\vec{A}_{\rm r}(\vec{r},t)}$ where: $\boxed{\vec{u}(t_{\rm r}) \equiv c\hat{\mathscr{r}} - \vec{v}(t_{\rm r})}$ and: $\boxed{\vec{B}_{\rm r}(\vec{r},t) = \frac{1}{c}\hat{\mathscr{r}}\times\vec{E}_{\rm r}(\vec{r},t)}$

$$\boxed{\vec{B}_{\rm r}(\vec{r},t) = \frac{1}{c}\frac{q}{4\pi\varepsilon_o}\frac{\mathscr{r}}{\left(\vec{\mathscr{r}}\cdot\vec{u}(t_{\rm r})\right)^3}\hat{\mathscr{r}}\times\left[\overbrace{\left(c^2 - v^2(t_{\rm r})\right)\vec{u}(t_{\rm r})}^{\text{term for generalized Coulomb field/velocity field}} + \overbrace{\vec{\mathscr{r}}\times\left(\vec{u}(t_{\rm r})\times\vec{a}(t_{\rm r})\right)}^{\text{term for radiation/acceleration field}}\right]}$$

Microscopically:

The ***acceleration*** {and/or ***deceleration***} of electric charges q and/or time-varying electric current densities $\left(e.g.\ \vec{J} = nq\vec{v};\ \partial\vec{J}/\partial t = nq\,\partial\vec{v}/\partial t \sim nq\vec{a}\right)$ "converts" (a portion of the) ***virtual*** photons (associated with the "static" ***Coulomb*** field, which individually have zero total energy/zero-frequency) to ***real*** photons (which individually have finite total energy/finite frequency *f*), which then freely propagate outward/away from the source of time-varying electric charge and/or electric current at the speed of light, c {in vacuum / free space}.

Since real photons individually carry energy/linear momentum/angular momentum, macroscopic *EM* waves carry energy/linear momentum angular momentum away from the source, in an irreversible manner – these *EM* waves propagate away from the source $\forall$ time. Energy/momentum ***must*** be input to the charged particle for this to happen – energy/momentum are {both} conserved in the radiation process.

{Note also that we can ***reverse*** the ***arrow of time*** $t \to -t$ in this process and thus learn about the ***absorption*** of energy/linear momentum/angular momentum by electric charges/currents from incoming/incident *EM* waves. . . .}

The total instantaneous power $P_{\rm r}\left(\vec{r},t\right)$ associated with ***radiation*** of *EM* waves from a source (assumed to be ***localized***) is obtained by integrating the retarded Poynting's vector $\vec{S}_{\rm r}\left(\vec{r},t\right)$ over a ***large*** spherical shell of radius $r \gg a$ = characteristic dimension of a localized source – this is known as the "far-field" limit, when $r \to \infty$:

$$\boxed{P_{\rm r}\left(\vec{r},t\right) = \oint_{S'} \vec{S}_{\rm r}\left(\vec{r}',t\right)\cdot d\vec{a}'_{\perp} = \frac{1}{\mu_o}\oint_{S'}\left(\vec{E}_{\rm r}\left(\vec{r}',t\right)\times\vec{B}_{\rm r}\left(\vec{r}',t\right)\right)\cdot d\vec{a}'_{\perp}}$$

The instantaneous power ***radiated*** is the limit of $P_{\rm r}\left(\vec{r},t\right)$ as $r \to \infty$: $\boxed{P_{\rm r}^{rad}\left(t\right) \equiv \lim_{r\to\infty} P_{\rm r}\left(\vec{r},t\right)}$

The physical reason for this definition is simple. In the so-called "near-zone", when $r \lesssim a$, the (generalized) Coulomb field(s) (microscopically consisting of ***virtual*** photons) are dominant in this region – thus, time-varying but ***non-radiating*** $\vec{E}$ and $\vec{B}$ ***fields*** are present in proximity to the source. The near-zone *EM* fields fall off/decrease/diminish as $\sim 1/\lambda\!\!\!^{-2}$ from the source.

In reality, for ***finite*** r, there is always a mixture of ***radiating*** and ***non-radiating*** *EM* fields present that is associated with any source. Expressed in a graphical manner in terms of r/a:

r/a

$$\underbrace{\ldots\ 10^{-4}\quad 10^{-3}\quad 10^{-2}\quad 10^{-1}\quad 10^{0}=1}_{\text{"near-zone" regime:}}\quad 10\quad \underbrace{100\quad 10^{3}\quad 10^{4}\quad 10^{5}\ldots\quad \infty}_{\text{"far-zone" regime}}$$

"near-zone" regime: $\left(\dfrac{r}{a}\lesssim 1\right)$ — Generalized Coulomb field(s) dominant (***virtual*** photons)

"far-zone" regime $\left(\dfrac{r}{a}\gg 1\right)$ — Radiation/acceleration field(s) dominant (freely propagating ***real*** photons)

The instantaneous *EM* power associated with the Generalized Coulomb field is:

$$\boxed{P_{\mathrm{r}}^{GCF}(\vec{r},t)=\oint_{S'}\vec{S}_{\mathrm{r}}^{GCF}(\vec{r}',t)\cdot d\vec{a}'_{\perp}=\frac{1}{\mu_o}\oint_{S'}\left(\vec{E}_{\mathrm{r}}^{GCF}(\vec{r}',t)\times\vec{B}_{\mathrm{r}}^{GCF}(\vec{r}',t)\right)\cdot d\vec{a}'}$$

But: $\boxed{\vec{E}_{\mathrm{r}}^{GCF}(\vec{r},t)\sim 1/\mathscr{r}^2}$ (even faster than this, if the ***net*** charge = 0, *e.g.* for higher order *EM* ***moments*** associated with ***electric*** dipoles, quadrupoles, octupoles, etc. …)

And: $\boxed{\vec{B}_{\mathrm{r}}^{GCF}(\vec{r},t)\sim 1/\mathscr{r}^2}$ (even faster than this, if the ***net*** charge = 0, *e.g.* for higher order *EM* ***moments*** associated with ***magnetic*** dipoles, quadrupoles, octupoles, etc. …)

$\Rightarrow$ $\boxed{\vec{S}_{\mathrm{r}}^{GCF}(\vec{r},t)\sim 1/\mathscr{r}^4}$ (ever faster, for high-order *EM* moments than a point charge distribution)

But: $\boxed{A_{\perp}^{\text{sphere}}=4\pi\mathscr{r}^2}$ = area of sphere of radius $\mathscr{r}$.

$\therefore$ $\boxed{P_{\mathrm{r}}^{GCF}(\vec{r},t)\sim\frac{1}{\mathscr{r}^4}\cdot\mathscr{r}^2\sim\frac{1}{\mathscr{r}^2}}$ $\Rightarrow$ *EM* power associated with Generalized Coulomb fields is only appreciable ***near*** the source.

Note that $\boxed{\lim_{r\to\infty}P_{\mathrm{r}}^{GCF}(\vec{r},t)=0}$ *i.e.* no *EM* power is associated with G.C.F. at $r=\infty$

$\Rightarrow$ "static" sources do not ***radiate*** *EM* energy.

On the other hand, the instantaneous *EM* power associated with the radiation/acceleration fields is:

$$\boxed{P_{\mathrm{r}}^{rad}(\vec{r},t)=\oint_{S'}\vec{S}_{\mathrm{r}}^{rad}(\vec{r}',t)\cdot d\vec{a}'_{\perp}=\frac{1}{\mu_o}\oint_{S'}\left(\vec{E}_{\mathrm{r}}^{rad}(\vec{r}',t)\times\vec{B}_{\mathrm{r}}^{rad}(\vec{r}',t)\right)\cdot d\vec{a}'_{\perp}}$$

But: $\boxed{\vec{E}_{\mathrm{r}}^{rad}\sim 1/\mathscr{r}}$ and $\boxed{B_{\mathrm{r}}^{rad}\sim 1/\mathscr{r}}$ $\Rightarrow$ $\boxed{\vec{S}_{\mathrm{r}}^{rad}(\vec{r},t)\sim 1/\mathscr{r}^2}$, $\boxed{A_{\perp}^{sphere}\sim\mathscr{r}^2}$

$\therefore$ $\boxed{P_{\mathrm{r}}^{rad}(\vec{r},t)\sim 1}$ (*i.e.* $P_{\mathrm{r}}^{rad}(\vec{r},t)$ is ***independent*** of the radius of the enclosing surface S')

Thus, we can simply pick $r\to\infty$ to ***eliminate*** the $P_{\mathrm{r}}^{GCF}(\vec{r},t)$ ***contribution***!!!

{*n.b.* for {unphysical} ***non-localized*** sources of time-varying *EM* radiation – *e.g.* ***infinite*** planes, ***infinitely*** long wires, ***infinite*** solenoids, *etc*. this requires a different approach altogether… }

In general, ***arbitrary*** configurations of ***localized***, time-dependent electric charge and/or electric current density distributions, $\boxed{\partial\rho(t_{\mathrm{r}})/\partial t_{\mathrm{r}}\equiv\dot{\rho}(t_{\mathrm{r}})}$ and $\boxed{\partial\vec{J}(t_{\mathrm{r}})/\partial t_{\mathrm{r}}\equiv\dot{\vec{J}}(t_{\mathrm{r}})}$ can/do produce *EM* radiation/freely-propagating *EM* waves.

As we learned in P435 (last semester), from the principal of linear superposition, we can always decompose an ***arbitrary*** electric charge and/or current distribution into a linear combination of *EM* moments of the electric charge/current distribution, *i.e.* electric monopole (electric charge), electric and magnetic dipole, electric and magnetic quadrupole, etc. … moments. This is true {separately} for both static and time-varying *EM* moments of the electric charge and/or current distribution(s).

For a ***point*** electric ***monopole*** field {E(0)}, *i.e.* $\boxed{\rho(\vec{r},t_r)=q\left(t_r-(t-\mathscr{r}/c)\right)\delta^3(\vec{r}')}$

$$\boxed{V_r^{(E0)}(\vec{r},t)=\frac{1}{4\pi\varepsilon_o}\int_{v'}\frac{\rho(\vec{r}',t_r)}{\mathscr{r}}d\tau'=\frac{q\left(t_r=(t-\mathscr{r}/c)\right)}{4\pi\varepsilon_o\mathscr{r}}}$$

Where $q(t_r)=$ ***total*** electric charge of the source at the ***retarded*** time t_r. But electric charge is (always) conserved, and furthermore, (by definition) a ***localized*** source is one that does ***not*** have electric charge q flowing into or away from it. Therefore, the electric monopole moment contribution/portion associated with the (retarded) potential(s) and *EM* fields is of necessity ***static*** – *i.e.* the electric monopole moment q has ***no*** *EM* ***radiation*** associated with it. In other words, there can be no ***net transversely*** polarized *EM* radiation emitted from a ***spherically-symmetric*** charge distribution! {See *e.g.* J. D. Jackson *Classical Electrodynamics* 3rd ed. *p*. 410 for additional/further details.}

The lowest-order ***electric*** multipole moment capable of producing *EM* radiation is that associated with a time-varying ***electric dipole moment*** $\boxed{\vec{p}(\vec{r}',t_r)=q\vec{d}(\vec{r}',t_r)}$ or: $\boxed{=q(\vec{r}',t_r)\vec{d}}$.

$\Rightarrow$ Electric dipole (E1) radiation originates from $\boxed{\dot{\rho}(\vec{r}',t_r)}$

The lowest-order ***magnetic*** multipole moment capable of producing *EM* radiation is that associated with a time-varying ***magnetic dipole moment*** $\boxed{\vec{m}(\vec{r}',t_r)=I\vec{a}(\vec{r}',t_r)}$ or: $\boxed{=I(\vec{r}',t_r)\vec{a}}$.

$\Rightarrow$ Magnetic dipole (M1) radiation originates from $\boxed{\dot{\vec{J}}(\vec{r}',t_r)}$

Each time-varying, localized, higher-order *EM* moment contributes in alternating ***succession*** between $\dot{\rho}(\vec{r}',t_r)$ and $\dot{\vec{J}}(\vec{r}',t_r)$ (*i.e.* electric *vs*. magnetic) multipole moment terms:

Time-varying localized electric moments: $\dot{\rho}(\vec{r}',t_r)$	**Time-varying localized magnetic moments:** $\dot{\vec{J}}(\vec{r}',t_r)$
$\dot{E}(0)$ electric monopole $\dot{q}$ **NO**!	
	$\dot{M}(0)$ magnetic monopole $\dot{g}$ **NO**! $\boxed{\exists\ \textit{no}\ \text{magnetic charges anyways…}}$
$\dot{E}(1)$ electric dipole $\dot{\vec{p}}=q\dot{\vec{d}}$	
	$\dot{M}(1)$ magnetic dipole $\dot{\vec{m}}=I\dot{\vec{a}}$
$\dot{E}(2)$ electric quadrupole $\dot{\overleftrightarrow{Q}}_E=2q\dot{\vec{d}}\vec{d}$	
	$\dot{M}(2)$ magnetic quadrupole $\dot{\overleftrightarrow{Q}}_M=2I\dot{\vec{a}}\vec{a}$
$\dot{E}(3)$ electric octupole	
	$\dot{M}(3)$ magnetic octupole
$\dot{E}(4)$ electric sextupole	
… *etc*…	$\dot{M}(4)$ magnetic sextupole

$\therefore$ We will consider/discuss the case of *EM* radiation from an oscillating E(1) electric dipole and then discuss case of radiation from an arbitrary localized source consisting of an arbitrary linear combination of time-varying *EM* moments, $\sum_{n=1}^{\infty}\left(a_n\dot{E}(n)+b_n\dot{M}(n)\right)$, where $\dot{E}(n)$ and $\dot{M}(n)$ are n^{th}-order time-varying electric and magnetic multipole moments, respectively.

E(1) Electric Dipole Radiation:

Consider an oscillating (*i.e.* harmonic/sinusoidally time-varying) electric dipole: $\boxed{\vec{p}(t)=q\vec{d}(t)}$ where the charge ***separation distance*** varies in time: $\boxed{\vec{d}(t)=\vec{d}(t)\hat{z}=d\cos(\omega t)\hat{z}}$, $\boxed{\omega=2\pi f}$

Then: $\boxed{\vec{p}(t)=qd\cos(\omega t)\hat{z}=p\cos(\omega t)\hat{z}}$, with: $\boxed{p=qd}$.

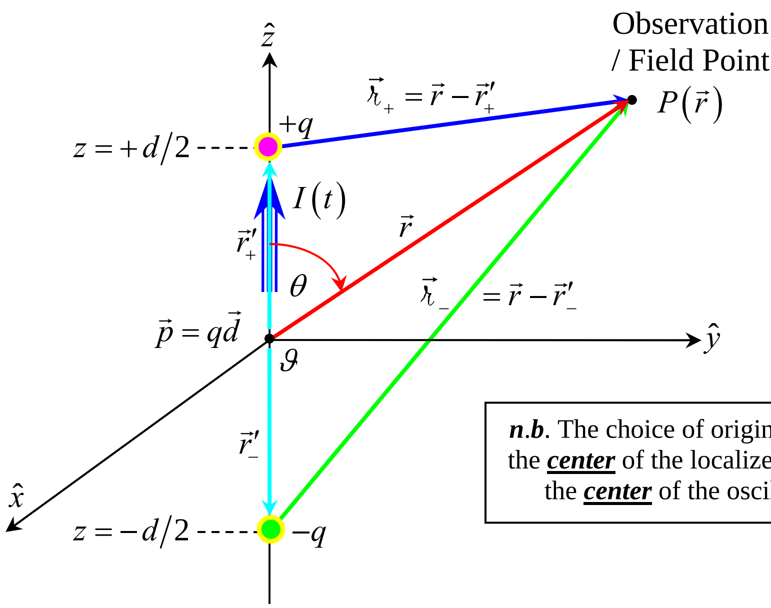
Equivalently, we can ***alternatively*** think of this as: $\boxed{\vec{p}(t)=q(t)\vec{d}}$, with: $\boxed{\vec{d}=d\hat{z}}$ = ***constant***, and with ***time-varying/oscillating*** electric charge: $\boxed{q(t)=q\cos(\omega t)}$.

Then: $\boxed{\vec{p}(t)=qd\cos(\omega t)\hat{z}=p\cos(\omega t)\hat{z}}$, with: $\boxed{p=qd}$. {*n.b.* same result!}

Either way one views/thinks about this, the physics associated with a harmonically time-varying/oscillating electric dipole moment $\boxed{\vec{p}(t)=p\cos(\omega t)\hat{z}=qd\cos(\omega t)\hat{z}}$ is unchanged.

n.b. $\exists$ an electric ***current*** associated with the oscillating electric dipole: $\boxed{\vec{I}(t)=\frac{dq(t)}{dt}\hat{z}}$, $\boxed{\vec{I}(t=0)=0}$

A picture of this, for a given moment/instant/snapshot in time $t=0$ is shown below:



n.b. The choice of origin is deliberately chosen at the ***center*** of the localized charge distribution – at the ***center*** of the oscillating electric dipole.

n.b. $\exists$ exist (as always) some ***subtleties*** associated with the calculation of the ***retarded*** potentials associated with ***moving*** point charges – we will address these subsequently, but not right here / right now… so, we'll stick with the oscillating charge $\boxed{q(t)=q\cos(\omega t)}$ version for now…

Now $\vec{p}(t) = p\cos(\omega t)\hat{z}$ refers to the time-dependence associated with itself. An observer at field point $P(\vec{r})$ at $\vec{r}$ "sees" the effects of the time-varying $\vec{p}(t)$ manifest themselves at a finite time later, $\boxed{t = t_{\mathrm{r}} + \imath/c}$ or: $\boxed{t_{\mathrm{r}} = t - \imath/c}$ due to the ***retarded*** nature of this problem.

Thus, $\vec{p}(t)$ used in the formulae for the ***retarded*** scalar and vector potentials ***must*** be evaluated at the ***retarded time*** t_{r}, *i.e.* $\boxed{\vec{p}(t) \to \vec{p}(t_{\mathrm{r}}) = q(t_{\mathrm{r}})\vec{d} = qd\cos(\omega t_{\mathrm{r}})\hat{z}}$.

$$\boxed{V_{\mathrm{r}}^{\mathrm{E}(1)}(\vec{r},t) = \underbrace{\frac{q(t_{\mathrm{r}}^{+})}{4\pi\varepsilon_o \imath_+}}_{\substack{\text{charge at}\\ +d/2\hat{z}}} - \underbrace{\frac{q(t_{\mathrm{r}}^{-})}{4\pi\varepsilon_o \imath_-}}_{\substack{\text{charge at}\\ -d/2\hat{z}}} = \underbrace{\left(\frac{q}{4\pi\varepsilon_o}\right)\frac{\cos(\omega t_{\mathrm{r}}^{+})}{\imath_+}}_{\substack{\text{charge at}\\ +d/2\hat{z}}} - \underbrace{\left(\frac{q}{4\pi\varepsilon_o}\right)\frac{\cos(\omega t_{\mathrm{r}}^{-})}{\imath_-}}_{\substack{\text{charge at}\\ -d/2\hat{z}}} = \left(\frac{q}{4\pi\varepsilon_o}\right)\left[\frac{\cos(\omega t_{\mathrm{r}}^{+})}{\imath_+} - \frac{\cos(\omega t_{\mathrm{r}}^{-})}{\imath_-}\right]}$$

$$\boxed{\vec{A}_{\mathrm{r}}^{\mathrm{E}(1)}(\vec{r},t) = \left(\frac{\mu_o}{4\pi}\right)\int \frac{I(t_{\mathrm{r}})}{\imath}\, d\vec{\ell}'} \quad \text{where: } \boxed{I(t_{\mathrm{r}}) = \frac{dq(t_{\mathrm{r}})}{dt} = -q\omega\sin(\omega t_{\mathrm{r}})} \quad \text{and: } \boxed{d\vec{\ell}' = dz\hat{z}}$$

Explicitly inserting the retarded time(s): $\boxed{t_{\mathrm{r}}^{\pm} = t - \imath_{\pm}/c}$:

$$\boxed{V_{\mathrm{r}}^{\mathrm{E}(1)}(r,t) = \left(\frac{q}{4\pi\varepsilon_o}\right)\left[\frac{\cos(\omega(t - \imath_+/c))}{\imath_+} - \frac{\cos(\omega(t - \imath_-/c))}{\imath_-}\right]}$$

$$\boxed{\vec{A}_{\mathrm{r}}^{\mathrm{E}(1)}(\vec{r},t) = -\left(\frac{\mu_o q\omega}{4\pi}\right)\int_{z=-d/2}^{z=+d/2} \frac{\sin[\omega(t - \imath/c)]}{\imath}\, dz\,\hat{z}}$$

Let us first focus our attention on calculating $V_{\mathrm{r}}^{\mathrm{E}(1)}(r,t)$. From the law of cosines {see P435 Lecture Notes 8 *r.e.* the derivation of the ***static*** multipole moment expansion}:

$$\boxed{\imath_{\pm} = \sqrt{r^2 \mp rd\cos\theta + (d/2)^2}}$$

However, we want to investigate *EM* radiation in the "far zone" where $r \gg d$.
For ***this*** situation we can make the following approximation:

$$\boxed{\imath_{\pm} = r\sqrt{1 \mp \left(\frac{d}{r}\right)\cos\theta + \cancel{\frac{1}{4}\left(\frac{d}{r}\right)^2}} \simeq r\sqrt{1 \mp \left(\frac{d}{r}\right)\cos\theta}} \quad \text{But: } \sqrt{1 \mp \in} \simeq 1 \mp \tfrac{1}{2}\in \text{ for } \in \ll 1.$$

Thus: $$\boxed{\imath_{\pm} \approx r\left(1 \mp \frac{1}{2}\left(\frac{d}{r}\right)\cos\theta\right) = r\left(1 \mp \left(\frac{d}{2r}\right)\cos\theta\right)} \quad \text{for } r \gg d.$$

Similarly/correspondingly:

$$\boxed{\frac{1}{\imath_{\pm}} \simeq \frac{1}{r\left(1 \mp \frac{1}{2}\left(\frac{d}{r}\right)\cos\theta\right)} \approx \frac{1}{r}\left(1 \pm \left(\frac{d}{2r}\right)\cos\theta\right)} \quad \text{for } r \gg d, \text{ since: } \frac{1}{1 \mp \in} \simeq 1 \pm \in \text{ for } \in \ll 1.$$

Likewise, for the $\cos\left(\omega\left(t-\mathscr{r}_\pm/c\right)\right)$ term, for the "far zone", when $r \gg d$ we have:

$$\boxed{\begin{aligned}\cos\left(\omega\left(t-\mathscr{r}_\pm/c\right)\right) &\approx \cos\left[\omega\left(t-\frac{r}{c}\left(1\mp\frac{d}{2r}\cos\theta\right)\right)\right] = \cos\left[\omega\left(t-\frac{r}{c}\right)\pm\left(\frac{\omega d}{2c}\right)\cos\theta\right] \\ &= \cos\left[\omega\left(t-\frac{r}{c}\right)\right]\cdot\cos\left[\left(\frac{\omega d}{2c}\right)\cos\theta\right]\mp\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\cdot\sin\left[\left(\frac{\omega d}{2c}\right)\cos\theta\right]\end{aligned}}$$

In order to proceed ***further***, we need to make an ***additional*** simplifying assumption, namely that the ***characteristic spatial dimension*** a of the ***source*** (***here***, $a = d$) is $\ll$ ***wavelength*** λ of the ***emitted radiation***, *i.e.* $d \ll \lambda$ $\{\lambda = c/f\}$. Thus we have: $d \ll c/f$ $\{f = \omega/2\pi\}$, or: $d \ll 2\pi c/\omega$ or: $d \ll c/\omega$.

n.b. This assumption is tantamount/physically equivalent to saying that we will neglect any/all ***time-retardation*** effects associated with finite *EM* propagation delay times over the ***dimensions characteristic of/associated*** with the ***source*** – *i.e.* changes in charge/current are essentially ***coherent/instantaneous*** over the {small} spatial dimensions of the source, relative to the wavelength of the emitted radiation.

Suppose we have a source (*e.g.* an ***atom***) with $a = d = 1\,nm = 10\,\mathring{A}$ emitting a $f = 1\ Hz$ sine-wave. Since *EM* radiation travels propagates at $1\,ft \simeq 30\,cm$ per nanosecond, a $1\,nm$ dimension source doesn't run into finite propagation decay time ***problems*** until:

$$c\Delta t \approx a = d\ (\textbf{here})\ \ i.e.\ \ c\Delta t \approx 1\,nm \Rightarrow \Delta t \simeq \frac{10^{-9}}{3\times10^{8}} \simeq 0.3\times10^{-17}\ \text{sec} \Rightarrow \underline{\underline{f \simeq 3\times10^{17}\ Hz}}$$

Thus, provided that we ***additionally are*** in the regime of $d \ll \lambda$, or $d \ll c/\omega$, *i.e.* $\boxed{\left(\frac{\omega d}{c}\right) \ll 1}$.

Noting that if: $\left(\frac{\omega d}{c}\right) \ll 1$, then: $\left|\left(\frac{\omega d}{2c}\right)\cos\theta\right| \ll 1\ \forall\ \theta\ \{0 \le \theta \le \pi\}$.

Then from the Taylor series expansions of $\cos(x) \approx 1$ and $\sin(x) \approx x$ for ***very*** small $x \ll 1$, we see that:

$$\boxed{\cos\left[\left(\frac{\omega d}{2c}\right)\cos\theta\right] \approx \cos(0) \simeq 1} \quad \text{and:} \quad \boxed{\sin\left[\left(\frac{\omega d}{2c}\right)\cos\theta\right] \approx \left(\frac{\omega d}{2c}\right)\cos\theta}$$

Thus: $$\boxed{\cos\left(\omega\left(t-\mathscr{r}_\pm/c\right)\right) \approx \cos\left[\omega\left(t-\frac{r}{c}\right)\right]\mp\left(\frac{\omega d}{2c}\right)\cos\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]}$$

Thus:

$$\boxed{\begin{aligned}V_{\mathrm{r}}^{\mathrm{E}(1)}(r,\theta,t) \approx \frac{q}{4\pi\varepsilon_o}\Bigg\{&\frac{1}{r}\left(1+\left(\frac{d}{2r}\right)\cos\theta\right)\left[\cos\left[\omega\left(t-\frac{r}{c}\right)\right]-\left(\frac{\omega d}{2c}\right)\cos\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right] \\ &-\frac{1}{r}\left(1-\left(\frac{d}{2r}\right)\cos\theta\right)\left[\cos\left[\omega\left(t-\frac{r}{c}\right)\right]+\left(\frac{\omega d}{2c}\right)\cos\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right]\Bigg\}\end{aligned}}$$

Expanding this out:

$$\boxed{\begin{aligned}V_{\rm r}^{\rm E(1)}(r,\theta,t) \approx \frac{q}{4\pi\varepsilon_o}\left(\frac{1}{r}\right)\Big\{ & \cancel{\cos\left[\omega\left(t-\frac{r}{c}\right)\right]} + \left(\frac{d}{2r}\right)\cos\theta\cos\left[\omega\left(t-\frac{r}{c}\right)\right] \\ & -\left(\frac{\omega d}{2c}\right)\cos\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right] - \cancel{\frac{\omega d^2}{4rc}\cos^2\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]} \\ & -\cancel{\cos\left[\omega\left(t-\frac{r}{c}\right)\right]} + \left(\frac{d}{2r}\right)\cos\theta\cos\left[\omega\left(t-\frac{r}{c}\right)\right] \\ & -\left(\frac{\omega d}{2c}\right)\cos\omega\sin\left[\omega\left(t-\frac{r}{c}\right)\right] + \cancel{\frac{\omega d^2}{4rc}\cos^2\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]}\Big\}\end{aligned}}$$

Thus:

$$\boxed{\begin{aligned}V_{\rm r}^{\rm E(1)}(r,\theta,t) &\approx \frac{q}{4\pi\varepsilon_o}\left(\frac{1}{r}\right)\left\{\left(\frac{d}{r}\right)\cos\theta\cos\left[\omega\left(t-\frac{r}{c}\right)\right] - \left(\frac{\omega d}{c}\right)\cos\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right\} \\ &= \frac{qd}{4\pi\varepsilon_o}\left(\frac{\cos\theta}{r}\right)\left\{\left(\frac{1}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right] - \left(\frac{\omega}{c}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}\end{aligned}}$$

But: $\boxed{p = |\vec{p}| = qd}$. Hence in the "far-zone" $d \ll r$ ***and*** $d \ll \lambda$:

$$\boxed{V_{\rm r}^{\rm E(1)}(r,\theta,t) \approx \frac{p}{4\pi\varepsilon_o}\left(\frac{\cos\theta}{r}\right)\left\{-\left(\frac{\omega}{c}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right] + \left(\frac{1}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}}$$

In the "far-zone" $d \ll r$, with the ***additional*** restriction that we've also imposed on the source *EM* radiation: $d \ll \lambda$. We now ***additionally*** require/impose a ***third*** restriction that the "far-zone" ***also*** be such that $\lambda \ll r$, thus we have the hierarchical relation: $\boxed{d \ll \lambda \ll r}$ for "far-zone" *EM* radiation, namely that for $\boxed{\lambda \ll r} \Rightarrow \boxed{\left(\frac{c}{\omega}\right) \ll r}$, then $\Rightarrow \boxed{\left(\frac{\omega}{c}\right) \gg \left(\frac{1}{r}\right)}$ *i.e.* $\boxed{\frac{1}{\lambda} \gg \frac{1}{r} \text{ for } \lambda \ll r}$.

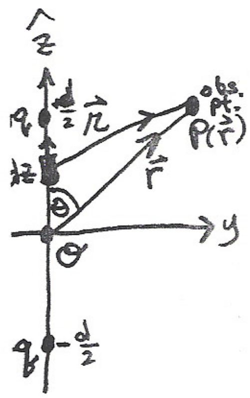
Thus for the far-zone, when $\boxed{d \ll \lambda \ll r}$ we can ***neglect*** the second term in the above expression for $V_{\rm r}^{\rm E(1)}(r,\theta,t)$.

Then: $\boxed{V_{\rm r}^{\rm E(1)}(r,\theta,t) \approx -\frac{p\omega}{4\pi\varepsilon_o c}\left(\frac{\cos\theta}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]}$ in the far-zone, for $\boxed{d \ll \lambda \ll r}$.

Note that in the ***static*** ***limit***, when $\omega \to 0$ it ***is*** necessary to ***retain*** the second term in the above expression; we obtain in this limit: $\boxed{V_{\rm r}^{\rm E(1)}(r,\theta) \approx -\frac{p}{4\pi\varepsilon_o}\left(\frac{\cos\theta}{r^2}\right)}$ {*cf* w/ P435 Lect. Notes – same!}

Now let us focus our attention on calculating $\vec{A}_{\text{r}}^{\text{E}(1)}(\vec{r},t)$:

$$\boxed{\vec{A}_{\text{r}}^{\text{E}(1)}(\vec{r},t) = -\left(\frac{\mu_o q\omega}{4\pi}\right)\int_{z=-d/2}^{z=+d/2}\frac{\sin\left[\omega(t-\mathscr{r}/c)\right]}{\mathscr{r}}dz\,\hat{z}}$$



Because the integration itself introduces a factor of d, then to ***first*** ***order*** in $(d/r)\ll 1$: $\boxed{\mathscr{r} = \sqrt{r^2 - 2rz\cos\theta + z^2} \approx r}$ with: $\boxed{|z| \le \frac{d}{2}}$

Thus: $\boxed{\int_{z=-d/2}^{z=+d/2}\frac{\sin\left[\omega(t-\mathscr{r}/c)\right]}{\mathscr{r}}dz \approx \frac{\sin\left[\omega(t-r/c)\right]}{r}d}$

Then: $\boxed{\vec{A}_{\text{r}}^{\text{E}(1)}(r,t) \approx -\frac{\mu_o(qd)\omega}{4\pi}\left(\frac{1}{r}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right)\hat{z}}$ but: $\boxed{p = qd}$

Thus: $\boxed{\vec{A}_{\text{r}}^{\text{E}(1)}(r,t) \approx -\frac{\mu_o p\omega}{4\pi}\left(\frac{1}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{z}}$

Note that in the ***static*** ***limit***, when $\omega \to 0$ then $\vec{A}_{\text{r}}^{\text{E}(1)}(r,t) \to 0$ as we expect.

Now that we have obtained the (retarded) scalar and vector potentials $V_{\text{r}}^{\text{E}(1)}(r,t)$ and $\vec{A}_{\text{r}}^{\text{E}(1)}(r,t)$ it is a "straight forward' exercise to compute the associated (retarded) EM fields, $\vec{E}_{\text{r}}^{\text{E}(1)}(r,t)$ and $\vec{B}_{\text{r}}^{\text{E}(1)}(r,t)$:

$$\boxed{\vec{E}_{\text{r}}^{\text{E}(1)}(\vec{r},t) = -\vec{\nabla}V_{\text{r}}^{\text{E}(1)}(\vec{r},t) - \frac{\partial\vec{A}_{\text{r}}^{\text{E}(1)}(\vec{r},t)}{\partial t}} \quad \text{and:} \quad \boxed{\vec{B}_{\text{r}}^{\text{E}(1)}(\vec{r},t) = \vec{\nabla}\times\vec{A}_{\text{r}}^{\text{E}(1)}(\vec{r},t)}$$

In ***spherical*** coordinates:

$$\vec{\nabla}V_{\text{r}}^{\text{E}(1)}(\vec{r},t) \approx \left[\frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\hat{\varphi}\right]\overbrace{\left\{\frac{-p\omega}{4\pi\varepsilon_o c}\frac{\cos\theta}{r}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right\}}^{\text{n.b. } V_{\text{r}}^{\text{E}(1)}(\vec{r},t) \text{ has no explicit } \hat{\varphi} \text{ dependence}}$$

$$= -\frac{p\omega}{4\pi\varepsilon_o c}\left\{\cos\theta\left(-\frac{1}{r^2}\sin\left[\omega\left(t-\frac{r}{c}\right)\right] - \frac{\omega}{rc}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\right)\hat{r} - \frac{1}{r^2}\sin\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}\right\}$$

$$= +\frac{p\omega}{4\pi\varepsilon_o cr}\left\{\left(\frac{\omega}{c}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\cos\theta\hat{r} + \left(\frac{1}{r}\right)\left[\cos\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r} + \sin\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}\right]\right\}$$

But for "far-zone" *EM* radiation, $\boxed{d \ll \lambda \ll r}$ we have: $\boxed{\frac{\omega}{c} \gg \frac{1}{r}}$

$$\therefore \quad \boxed{\left(\frac{\omega}{c}\right)\underbrace{\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\cos\theta}_{\sim\vartheta(1)} \gg \left(\frac{1}{r}\right)\left\{\underbrace{\cos\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]}_{\sim\vartheta(1)} + \underbrace{\sin\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]}_{\sim\vartheta(1)}\right\}}$$

So we can neglect/drop the $\boxed{\left(\frac{1}{r}\right)\left\{\left[\cos\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r}+\sin\theta\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}\right]\right\}}$ terms.

$$\therefore \quad \boxed{\vec{\nabla}V_{\text{r}}^{\text{E}(1)}(\vec{r},t)\approx+\frac{p\omega^2\cos\theta}{4\pi\varepsilon_o c^2 r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r}}$$

And: $\boxed{\frac{\partial\vec{A}_{\text{r}}^{\text{E}(1)}(\vec{r},t)}{\partial t}\approx-\frac{\mu_o p\omega}{4\pi}\left(\frac{1}{r}\right)\frac{\partial}{\partial t}\left(\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\right)\hat{z}=-\frac{\mu_o p\omega^2}{4\pi r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{z}}$

But: $\boxed{\hat{z}=\cos\theta\hat{r}-\sin\theta\hat{\theta}}$ in spherical coordinates.

$$\therefore \quad \boxed{\frac{\partial\vec{A}_{\text{r}}^{\text{E}(1)}(\vec{r},t)}{\partial t}\approx-\frac{\mu_o p\omega^2}{4\pi r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\left[\cos\theta\hat{r}-\sin\theta\hat{\theta}\right]}$$

Then for far-zone *EM* radiation, with $\boxed{d\ll\lambda\ll r}$: $\boxed{\vec{E}_{\text{r}}^{\text{E}(1)}(\vec{r},t)=-\vec{\nabla}V_{\text{r}}^{\text{E}(1)}(\vec{r},t)-\frac{\partial\vec{A}_{\text{r}}^{\text{E}(1)}(\vec{r},t)}{\partial t}}$

$$\boxed{\vec{E}_{\text{r}}^{\text{E}(1)}(\vec{r},t)\approx-\frac{p\omega^2}{4\pi\varepsilon_o c^2 r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\cos\theta\hat{r}+\frac{\mu_o p\omega^2}{4\pi r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\left[\cos\theta\hat{r}-\sin\theta\hat{\theta}\right]}$$

But: $\boxed{c^2=\frac{1}{\varepsilon_o\mu_o}}$ or: $\boxed{\frac{1}{c^2}=\varepsilon_o\mu_o}$

$$\therefore \quad \boxed{\begin{aligned}\vec{E}_{\text{r}}^{\text{E}(1)}(\vec{r},t)\approx&\cancel{-\frac{\mu_o p\omega^2}{4\pi r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]}\cos\theta\hat{r}+\cancel{\frac{\mu_o p\omega^2}{4\pi r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]}\cos\theta\hat{r}\\&-\frac{\mu_o p\omega^2}{4\pi r}\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\sin\theta\hat{\theta}\end{aligned}}$$

Or: $\boxed{\vec{E}_{\text{r}}^{\text{E}(1)}(\vec{r},t)\approx-\frac{\mu_o p\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}}$

Then: $\boxed{\vec{B}_{\text{r}}^{\text{E}(1)}(\vec{r},t)=\vec{\nabla}\times\vec{A}_{\text{r}}^{\text{E}(1)}(\vec{r},t)}$

with: $\boxed{\vec{A}_{\text{r}}^{\text{E}(1)}(r,t)\approx-\frac{\mu_o p\omega}{4\pi}\left(\frac{1}{r}\right)\sin\left[\omega\left(t-{}^{r}\!/\!{}_{c}\right)\right]\hat{z}=-\frac{\mu_o p\omega}{4\pi}\left(\frac{1}{r}\right)\sin\left[\omega\left(t-{}^{r}\!/\!{}_{c}\right)\right]\left[\cos\theta\hat{r}-\sin\theta\hat{\theta}\right]}$

Thus:

$$\boxed{\vec{B}_{\text{r}}^{\text{E}(1)}(\vec{r},t)=\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\overset{=0}{\cancel{A_\varphi}}\right)-\overset{=0}{\cancel{\frac{\partial A_\theta}{\partial\varphi}}}\right]\hat{r}+\frac{1}{r}\left[\frac{1}{\sin\theta}\overset{=0}{\cancel{\frac{\partial A_r}{\partial\varphi}}}-\frac{\partial}{\partial r}\left(r\overset{=0}{\cancel{A_\varphi}}\right)\right]\hat{\theta}+\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta)-\frac{\partial A_r}{\partial\theta}\right]\hat{\varphi}}$$

Thus:

$$\boxed{\begin{aligned}\vec{B}_{\rm r}^{\rm E(1)}(\vec{r},t) &= \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta)-\frac{\partial A_r}{\partial\theta}\right]\hat{\varphi}\\ &\approx\left(\frac{1}{r}\right)\left[-\frac{\mu_o p\omega}{4\pi}\right]\left\{\frac{\partial}{\partial r}\left(\frac{1}{\cancel{r}}\cdot\cancel{r}\sin\left[\omega\left(t-\frac{r}{c}\right)\right](-\sin\theta)\right)-\frac{1}{r}\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\frac{\partial\cos\theta}{\partial\theta}\right\}\hat{\varphi}\\ &=-\frac{\mu_o p\omega}{4\pi r}\left\{+\left(\frac{\omega}{c}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\sin\theta+\left(\frac{1}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]\sin\theta\right\}\hat{\varphi}\\ &=-\frac{\mu_o p\omega}{4\pi r}\left\{\left(\frac{\omega}{c}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]+\underbrace{\left(\frac{1}{r}\right)\sin\left[\omega\left(t-\frac{r}{c}\right)\right]}_{\therefore\ \text{neglect}}\right\}\sin\theta\hat{\varphi}\end{aligned}}$$

Again, $\boxed{\left(\frac{\omega}{c}\right)\gg\left(\frac{1}{r}\right)}$ here, because $\boxed{\lambda\ll r}$, thus

$$\boxed{\vec{B}_{\rm r}^{\rm E(1)}(\vec{r},t)\approx-\frac{\mu_o p\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}}\ \text{and:}\ \boxed{\vec{E}_{\rm r}^{\rm E(1)}(\vec{r},t)\approx-\frac{\mu_o p\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}}$$

Now since $\boxed{\hat{r}\times\hat{\theta}=\hat{\varphi}}$, once again we see that: $\boxed{\vec{B}_{\rm r}^{\rm E(1)}(\vec{r},t)=\frac{1}{c}\hat{r}\times\vec{E}_{\rm r}^{\rm E(1)}(\vec{r},t)}$, *i.e.* $\vec{B}\perp\vec{E}$ and $\vec{B}\perp\hat{r}$

Note also that:

a.) $\vec{E}_{\rm r}^{\rm E(1)}$ and $\vec{B}_{\rm r}^{\rm E(1)}$ both vary as $\sim 1/r$.

b.) $\vec{E}_{\rm r}^{\rm E(1)}(\vec{r},t)$ and $\vec{B}_{\rm r}^{\rm E(1)}(\vec{r},t)$ are ***in-phase*** with each other.

c.) $\vec{E}_{\rm r}^{\rm E(1)}(\vec{r},t)$ and $\vec{B}_{\rm r}^{\rm E(1)}(\vec{r},t)$ have the ***same*** angular dependence ($\sim\sin\theta$).

The *EM* ***radiation*** energy density, $u_{\rm E(1)}^{rad}(\vec{r},t)$ associated with the oscillating E(1) electric dipole for far-zone *EM* radiation $\{d\ll\lambda\ll r\}$ is:

$$\boxed{\begin{aligned}u_{\rm E(1)}^{rad}(\vec{r},t) &= u_{\rm E(1)}^{Erad}(\vec{r},t)+u_{\rm E(1)}^{Mrad}(\vec{r},t)=\frac{1}{2}\left(\varepsilon_o\vec{E}_{\rm r}^{\rm E(1)}(\vec{r},t)\cdot\vec{E}_{\rm r}^{\rm E(1)}(\vec{r},t)+\frac{1}{\mu_o}\vec{B}_{\rm r}^{\rm E(1)}(\vec{r},t)\cdot\vec{B}_{\rm r}^{\rm E(1)}(\vec{r},t)\right)\\ &\approx\frac{1}{2}\left(\frac{\varepsilon_o\mu_o^2p^2\omega^4}{16\pi^2}\left(\frac{\sin^2\theta}{r^2}\right)\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]+\frac{\mu_o^{\cancel{2}}p^2\omega^4}{\cancel{\mu_o}16\pi^2c^2}\left(\frac{\sin^2\theta}{r^2}\right)\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]\right)\\ &=\frac{1}{2}\left(\frac{\mu_o p^2\omega^4}{16\pi^2c^2}\left(\frac{\sin^2\theta}{r^2}\right)\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]+\frac{\mu_o p^2\omega^4}{16\pi^2c^2}\left(\frac{\sin^2\theta}{r^2}\right)\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]\right)\end{aligned}}\left(\frac{Joules}{m^3}\right)$$

n.b. $\boxed{u_{\rm E(1)}^{Erad}(\vec{r},t)=u_{\rm E(1)}^{Mrad}(\vec{r},t)}$ using: $\boxed{c^2=1/\varepsilon_o\mu_o}$ or: $\boxed{\varepsilon_o=1/\mu_o c^2}$.

$$\therefore\ \boxed{u_{\rm E(1)}^{rad}(\vec{r},t)\approx\frac{\mu_o p^2\omega^4}{16\pi^2c^2}\left(\frac{\sin^2\theta}{r^2}\right)\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]}\left(\frac{Joules}{m^3}\right)$$ for: $\boxed{d\ll\lambda\ll r}$ "far zone" limit

The *EM* energy radiated by an oscillating electric dipole, in the "far zone" $\{d \ll \lambda \ll r\}$ limit is given by Poynting's vector:

$$\boxed{\vec{S}^{rad}_{\mathrm{E(1)}}(\vec{r},t) = \frac{1}{\mu_o}\left(\vec{E}^{\mathrm{E(1)}}_{\mathrm{r}}(\vec{r},t) \times \vec{B}^{\mathrm{E(1)}}_{\mathrm{r}}(\vec{r},t)\right)}$$

$$\boxed{\begin{array}{l} \hat{r}\times\hat{\varphi} = \hat{\varphi} \\ \hat{\theta}\times\hat{\varphi} = \hat{r} \\ \hat{\varphi}\times\hat{r} = \hat{\theta} \end{array}}$$

$$\boxed{\vec{S}^{rad}_{\mathrm{E(1)}}(\vec{r},t) \approx \left(\frac{1}{\cancel{\mu_o}}\right)\left(\frac{\cancel{\mu_o} p\omega^2}{4\pi r}\right)\left(\frac{\mu_o p\omega^2}{4\pi rc}\right)\sin^2\theta\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]\underbrace{\left[\hat{\theta}\times\hat{\varphi}\right]}_{=+\hat{r}}}$$

Or: $$\boxed{\vec{S}^{rad}_{\mathrm{E(1)}}(\vec{r},t) \approx \frac{\mu_o p^2\omega^4}{16\pi^2 c}\left(\frac{\sin^2\theta}{r^2}\right)\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{r}} \quad \left(\frac{Watts}{m^2}\right)$$

$\Rightarrow$ Radial outward flow of *EM* field energy for: $\boxed{d \ll \lambda \ll r}$ "far zone" limit

The *EM* radiation ***linear*** momentum density associated with an oscillating electric dipole, in the far zone $\{d \ll \lambda \ll r\}$ is given by:

$$\boxed{\vec{\wp}^{rad}_{\mathrm{E(1)}}(\vec{r},t) = \mu_o\varepsilon_o\vec{S}^{rad}_{\mathrm{E(1)}}(\vec{r},t) = \frac{1}{c^2}\vec{S}^{rad}_{\mathrm{E(1)}}(\vec{r},t)}$$

Or: $$\boxed{\vec{\wp}^{rad}_{\mathrm{E(1)}}(\vec{r},t) \simeq \frac{\mu_o p^2\omega^4}{16\pi^2 r^2 c^3}\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]\sin^2\theta\hat{r}} \quad \left(\frac{kg}{m^2\text{-sec}}\right)$$

$\Rightarrow$ Radial outward *EM* field linear momentum for: $\boxed{d \ll \lambda \ll r}$ "far zone" limit

The *EM* radiation ***angular*** momentum density associated with an oscillating electric dipole, in the far zone $\{d \ll \lambda \ll r\}$ is given by:

$$\boxed{\vec{\ell}^{rad}_{\mathrm{E(1)}}(\vec{r},t) = \vec{r}\times\vec{\wp}^{rad}_{\mathrm{E(1)}}(\vec{r},t)}$$

$$\boxed{\vec{\ell}^{rad}_{\mathrm{E(1)}}(r,t) \simeq \frac{\mu_o p^2\omega^4}{16\pi^2 r^2 c^3}\cos^2\left[\omega\left(t-\frac{r}{c}\right)\right]\sin^2\theta\left(\hat{r}\times\hat{r}\right) \equiv 0} \quad \left(\frac{kg}{m\text{-sec}}\right)$$

$\Rightarrow$ ***No*** *EM* field angular momentum for: $\boxed{d \ll \lambda \ll r}$ "far zone" limit

n.b. The ***exact*** $\vec{\ell}^{rad}_{\mathrm{E(1)}}(\vec{r},t) \not\equiv 0$ *i.e.* ignore ***restrictions*** on far-zone limit, keep ***all*** higher-order terms . . . we have neglected $\vec{E}^{\mathrm{E(1)}}_{\mathrm{r}} \sim \hat{r}$ term which is non-negligible in the ***near-zone*** $(d \sim r)$ and also in the so-called ***intermediate***, or ***inductive zone*** $(\lambda \sim r)$.

Time-Averaged Quantities for E(1) Radiation from an Oscillating Electric Dipole:

Recall the definition of time average: $\boxed{\langle A(t)\rangle \equiv \frac{1}{\tau}\int_{t=0}^{t=\tau} A(t)\,dt = \frac{1}{\tau}\int_{t=0}^{t=\tau} A_o \cos^2 \omega t\, dt = \frac{1}{2}A_o}$

The time-averaged *EM* radiation energy density associated with an oscillating electric dipole is:

$$\boxed{\left\langle u_{\mathrm{E}(1)}^{rad}(\vec{r},t)\right\rangle \approx \left(\frac{\mu_o p^2\omega^4}{32\pi^2 c^2}\right)\left(\frac{\sin^2\theta}{r^2}\right)} \left(\frac{Joules}{m^3}\right) \quad \text{for: } \boxed{d \ll \lambda \ll r} \text{ “far-zone” limit}$$

The time-averaged |Poynting's vector|, which is also the ***intensity*** $I_{\mathrm{E}(1)}^{rad}$ of *EM* radiation associated with an oscillating electric dipole is:

$$\boxed{I_{\mathrm{E}(1)}^{rad}(\vec{r}) \equiv \left\langle \left|\vec{S}_{\mathrm{E}(1)}^{rad}(\vec{r},t)\right|\right\rangle = \frac{1}{2}c\varepsilon_o\left\langle \left(E_{\mathrm{r}}^{\mathrm{E}(1)}(\vec{r},t)\right)^2\right\rangle \approx \left(\frac{\mu_o p^2\omega^4}{32\pi^2 c}\right)\left(\frac{\sin^2\theta}{r^2}\right)} \left(\frac{Watts}{m^2}\right) \quad \text{for: } \boxed{d \ll \lambda \ll r \text{ “far-zone” limit}}$$

We also see that: $\boxed{I_{\mathrm{E}(1)}^{rad}(\vec{r}) \equiv \left\langle \left|\vec{S}_{\mathrm{E}(1)}^{rad}(\vec{r},t)\right|\right\rangle = c\left\langle u_{\mathrm{E}(1)}^{rad}(\vec{r},t)\right\rangle} \left(\frac{Watts}{m^2}\right)$.

The time-averaged *EM* radiated power associated with an oscillating electric dipole is:

$$\boxed{\begin{aligned}\left\langle P_{\mathrm{E}(1)}^{rad}(\vec{r},t)\right\rangle &= \int_S \left\langle \vec{S}_{\mathrm{E}(1)}^{rad}(\vec{r},t)\right\rangle \cdot d\vec{a}_\perp \approx \frac{\mu_o p^2\omega^4}{32\pi^2 c \cancel{r^2}}\int_{\theta=0}^{\theta=\pi}\int_{\varphi=0}^{\varphi=2\pi} \cancel{r^2}\sin^2\theta \underbrace{\sin\theta d\theta}_{=d\cos\theta} d\varphi \\ &\simeq \frac{\mu_o p^2\omega^4}{{}_{16}\cancel{32}\pi^{\cancel{2}} c^2}\cancel{2}\pi\int_{\theta=0}^{\theta=\pi}\sin^2\theta\, d\cos\theta = \frac{\mu_o p^2\omega^4}{16\pi c^2}\int_{\theta=0}^{\theta=\pi}\sin^2\theta\, d\cos\theta\end{aligned}}$$

Let: $\boxed{u=\cos\theta}$, $\boxed{du=-d(\cos\theta)}$, $\boxed{\theta=0 \Rightarrow u=1}$, $\boxed{\theta=\pi \Rightarrow u=-1}$, $\boxed{\sin^2\theta = 1-\cos^2\theta = 1-u^2}$

$$\therefore \quad \boxed{\int_{-1}^{+1}(1-u^2)\,du = \left(u-\frac{1}{3}u^3\right)\Bigg|_{-1}^{+1} = +1-\frac{1}{3}+1-\frac{1}{3} = 2-\frac{2}{3} = \frac{4}{3}}$$

$\therefore$ The time-averaged radiated power associated with an oscillating electric dipole is:

$$\boxed{\left\langle P_{\mathrm{E}(1)}^{rad}(\vec{r},t)\right\rangle \approx \left(\frac{\mu_o p^2\omega^4}{12\pi c}\right)} (Watts) \text{ for: } \boxed{d \ll \lambda \ll r} \text{ “far-zone” limit}$$

n.b. $\left\langle P_{\mathrm{E}(1)}^{rad}(\vec{r},t)\right\rangle$ has ***no*** *r*-dependence!

Note that time-averaged radiated power varies as the 4th power of frequency!

The time-averaged *EM* radiation ***linear*** momentum density associated with an oscillating electric dipole is:

$$\boxed{\left\langle \vec{\wp}_{\mathrm{E}(1)}^{rad}(\vec{r},t)\right\rangle = \frac{1}{c^2}\left\langle \vec{S}_{\mathrm{E}(1)}^{rad}(\vec{r},t)\right\rangle = \frac{1}{c}\left\langle u_{\mathrm{E}(1)}^{rad}(\vec{r},t)\right\rangle \hat{r} \approx \left(\frac{\mu_o p^2\omega^4}{32\pi^2 c^3}\right)\left(\frac{\sin^2\theta}{r^2}\right)\hat{r}} \left(\frac{kg}{m^2\text{-}sec}\right) \text{ for: } \boxed{d \ll \lambda \ll r \text{ “far-zone” limit}}$$

The time-averaged *EM* radiation ***angular*** momentum density associated with an oscillating electric dipole is:

$$\boxed{\left\langle \vec{\ell}^{\,rad}_{\mathrm{E(1)}}(\vec{r},t)\right\rangle = \vec{r}\times\left\langle \vec{\wp}^{\,rad}_{\mathrm{E(1)}}(\vec{r},t)\right\rangle \approx \left(\frac{\mu_o p^2\omega^4}{32\pi^2 c^3}\right)\left(\frac{\sin^2\theta}{r}\right)(\hat{r}\times\hat{r}) \equiv 0}\ \left(\frac{kg}{m\text{-sec}}\right) \quad \text{for:} \quad \boxed{\begin{array}{c} d \ll \lambda \ll r \\ \text{“far-zone” limit}\end{array}}$$

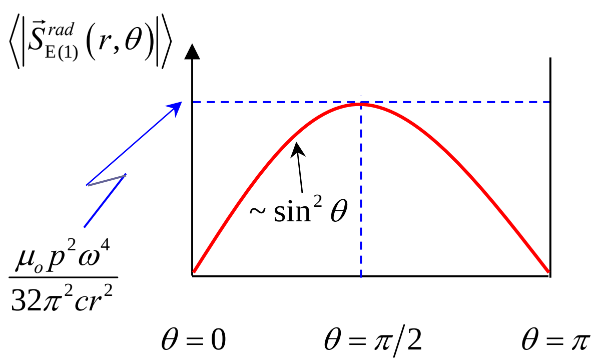
n.b. The ***exact*** $\left\langle \vec{\ell}^{\,rad}_{\mathrm{E(1)}}(\vec{r})\right\rangle \not\equiv 0$ *i.e.* ignore ***restrictions*** on far-zone limit, keep ***all*** higher-order terms . . . we have neglected the $\vec{E}^{\mathrm{E(1)}}_{\mathrm{r}} \sim \hat{r}$ term which is non-negligible in the ***near-zone*** $(d \sim r)$ and also in the so-called ***intermediate***, or ***inductive zone*** $(\lambda \sim r)$.

Note that because: $\boxed{I^{rad}_{\mathrm{E(1)}}(\vec{r}) \equiv \left\langle \left|\vec{S}^{rad}_{\mathrm{E(1)}}(\vec{r},t)\right|\right\rangle \approx \left(\frac{\mu_o p^2\omega^4}{32\pi^2 c}\right)\left(\frac{\sin^2\theta}{r^2}\right)}\ \left(\frac{Watts}{m^2}\right)$

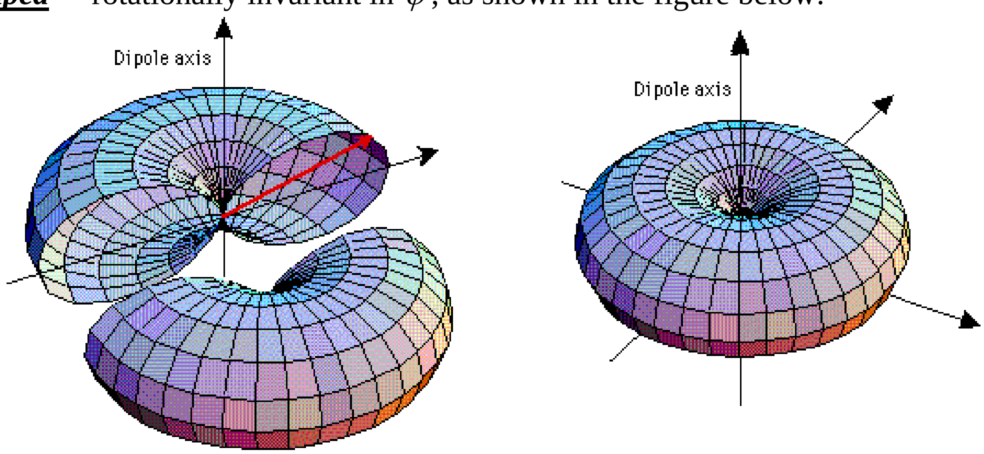
$\Rightarrow \boxed{\left\langle \vec{S}^{rad}_{\mathrm{E(1)}}(r,\theta=0,\varphi)\right\rangle = \left\langle \vec{S}^{rad}_{\mathrm{E(1)}}(r,\theta=\pi,\varphi)\right\rangle = 0}$ since: $\boxed{\sin^2 0 = \sin^2\pi = 0}$

i.e. ***no*** *EM* radiation occurs along the ***axis*** of the electric dipole ($\hat{z}$ axis)

EM radiation for E(1) electric dipole is peaked/maximum at $\theta = \pi/2$ (then $\sin^2\theta = 1$)
i.e. maximum *EM* radiation occurs $\perp$ to the axis of the electric dipole:



Thus, the intensity profile $I^{rad}_{\mathrm{E(1)}}(\vec{r})$ in 3-D {for fixed r} for E(1) electric dipole radiation is ***donut-shaped*** - rotationally invariant in φ, as shown in the figure below:



<u>Griffiths Example 11.1</u>:

The time-averaged power for E(1) electric dipole radiation is $\boxed{\left\langle P_{\mathrm{E}(1)}^{rad}\right\rangle \approx \frac{\mu_o p^2 \omega^4}{12\pi c}}$.

Note that $\left\langle P_{\mathrm{E}(1)}^{rad}\right\rangle \sim \omega^4$ (or $\sim f^4$, or $\sim \lambda^{-4}$)

For red light: $\boxed{\lambda_{\text{red}} \simeq 780\, nm} \Rightarrow \boxed{f_{\text{red}} = \frac{c}{\lambda_{\text{red}}} = \frac{3\times 10^8}{780\times 10^{-9}} \simeq 3.85\times 10^{14}\, Hz}$

For violet light: $\boxed{\lambda_{\text{violet}} \simeq 350\, nm} \Rightarrow \boxed{f_{\text{violet}} = \frac{c}{\lambda_{\text{violet}}} = \frac{3\times 10^8}{350\times 10^{-9}} \simeq 8.57\times 10^{14}\, Hz}$

Hence: $\boxed{\left(\frac{\left\langle P_{\mathrm{E}(1)}^{\text{violet}}\right\rangle}{\left\langle P_{\mathrm{E}(1)}^{\text{red}}\right\rangle}\right) = \left(\frac{f_{\text{violet}}}{f_{\text{red}}}\right)^4 = \left(\frac{8.57\times 10^{14}}{3.85\times 10^{14}}\right)^4 = (2.23)^4 \simeq 24.67}$.

$\Rightarrow \left\langle P_{\mathrm{E}(1)}^{rad}\right\rangle \sim \omega^4$ explains why the sky is blue! Sunlight {unpolarized light} incident on O_2 & N_2 molecules in the earth's atmosphere stimulates the O & N atoms – vibrates the {bound} atomic electrons at {angular} frequency ω, causing them to oscillate as electric dipoles! Solar *EM* radiation at a given angular frequency ω is thus absorbed and re-emitted in this *EM* radiation + atom ***<u>scattering</u>*** process.

The above formula for *EM* power radiated as E(1) electric dipole radiation by such atoms, by time-reversal invariance of the *EM* interaction, is also the *EM* power absorbed by atoms, thus we see that because of the ω^4-dependence of $\left\langle P_{\mathrm{E}(1)}^{rad}\right\rangle$, the higher frequency/shorter wavelength radiation (*i.e.* blue/violet light) is preferentially scattered ***<u>much</u>*** ***<u>more</u>*** ***<u>so</u>*** than the lower frequency/longer wavelength radiation (*i.e.* red light).

The Earth's sky appears blue {*e.g.* to an observer on the ground, or even *e.g.* a space shuttle astronaut in orbit around the earth} because the light from the sky is ***<u>scattered</u>*** (*i.e.* ***<u>re-radiated</u>***) ***<u>light</u>***, which is preferentially in the blue/violet portion of the visible light *EM* spectrum. The scattering of *EM* radiation off of atoms is known as ***<u>Rayleigh</u>*** ***<u>scattering</u>***.

Note that precisely same physics also simultaneously explains why the Sun appears ***<u>red</u>*** *e.g.* to an observer on the ground at ***<u>sunrise</u>*** and ***<u>sunset</u>*** – because at these times of the day, path that the sunlight takes through the atmosphere is the longest, relative to that associated *e.g.* with its position at {local} noon. If the higher-frequency blue/violet light is preferentially scattered ***<u>out</u>*** of the beam of sunlight, what is left ***<u>in</u>*** the beam of sunlight ***<u>after</u>*** traversing the entire thickness of the Earth's atmosphere is the lower-frequency, orange-red light.

Note that the Sun is a ***black-body radiator*** – its *EM* spectrum peaks in the infra-red region – thus it is ***NOT*** flat by any means {also is affected by frequency-dependent absorption in the atmosphere}:

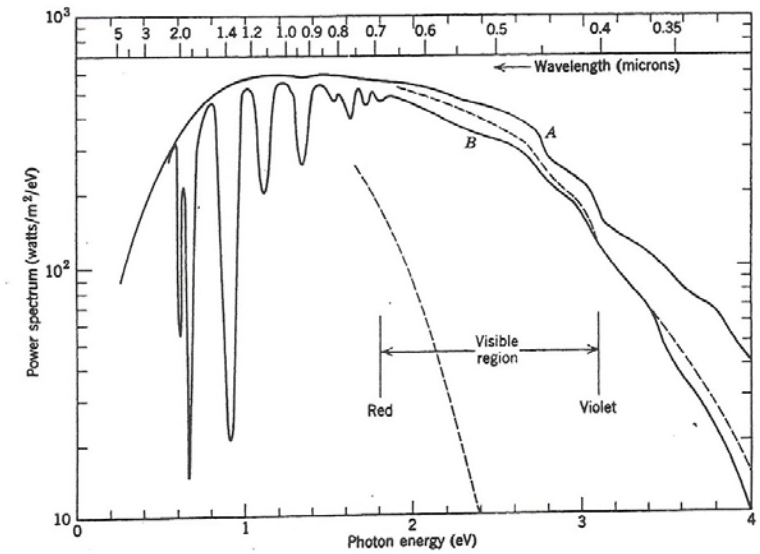
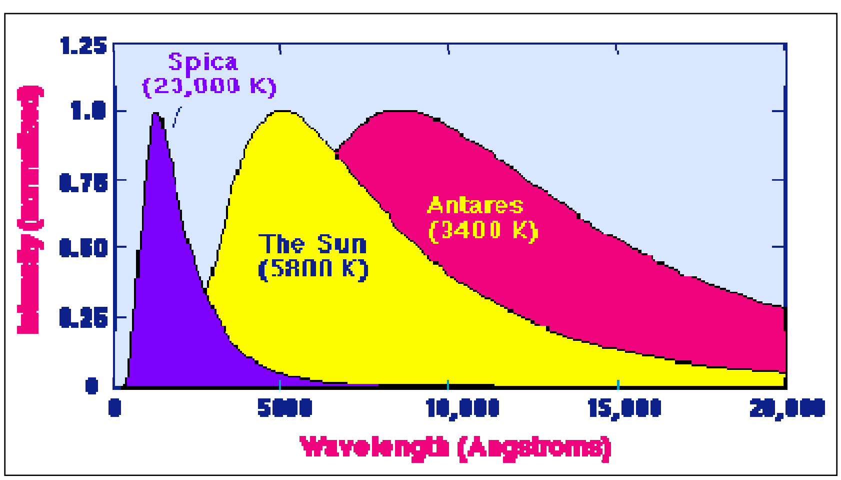


Figure 10.4 Power spectrum of solar radiation (in watts per square meter per electron volt) as a function of photon energy (in electron volts). Curve *A* is the incident spectrum above the atmosphere. Curve *B* is a typical sea-level spectrum with the sun at the zenith. The absorption bands below 2 eV are chiefly from water vapor and vary from site to site and day to day. The dashed curves give the expected sea-level spectrum at zenith and at sunrise-sunset if the only attenuation is from Rayleigh scattering by a dry, clean atmosphere.

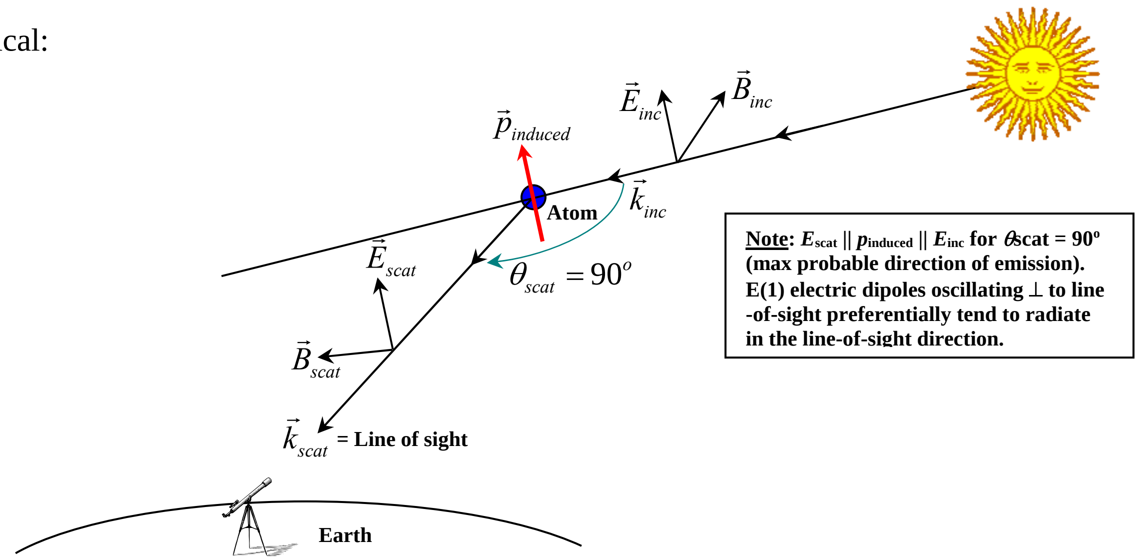
Note the log scale on the vertical axis! Thus, there is not much violet light in the Sun's *EM* spectrum, and hence there is a delicate "balancing" act of flux of *EM* radiation from the Sun {convoluted} with its black-body spectrum and the scattering of this radiation by atoms in the Earth's atmosphere – thus we see the sky as blue. Thus, if the black-body temperature of the sun was different, then the color of the Earth's sky in the visible portion of the *EM* spectrum would also be different – compare the black-body spectra of our Sun *e.g.* with that of Spica (260 *ly* away in the Virgo constellation) and Antares (a red giant 600 *ly* away in the Scorpio constellation):



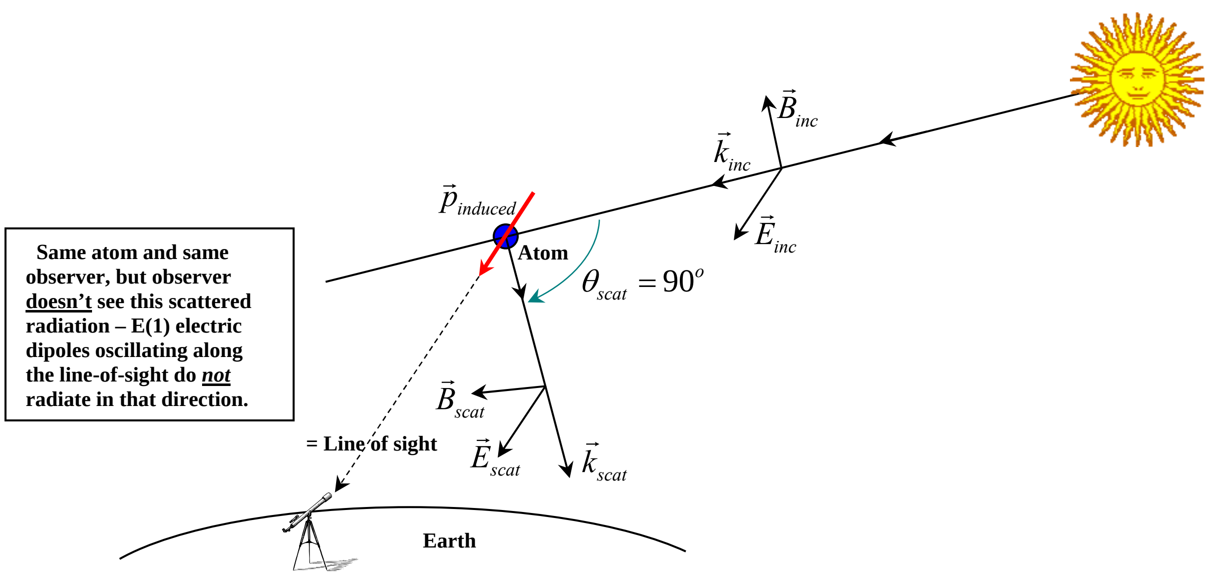
Light from the Sun is unpolarized (*i.e.* it consists of all polarizations, randomly oriented over time). However, because *EM* waves are ***transversely*** polarized (defined by the orientation of the $\vec{E}$-field vector) an incident *EM* plane wave from the Sun with polarization in a given direction ($\perp$ to $\vec{k}$-propagation direction) will (transitorily) induce electric dipole moments in gas atoms in earth's atmosphere, via $\vec{p}_{mol}(\omega)=\alpha_{mol}(\omega)\vec{E}_{inc}$, where $\alpha_{mol}(\omega)$ is the ***molecular polarizability*** at {angular} frequency ω {see P435 Lect. Notes 12 and P436 Lect. Notes 7.5}.

The axis of induced electric dipole moments will be || to the plane of polarization of incident wave at that instant, hence the scattered radiation emitted by the atom will be preferentially at $\theta = 90^\circ = \pi/2$ (*i.e.* $\perp$) to the axis of the (induced) electric dipole of gas atoms in earth's atmosphere. There are two specific/limiting cases to consider – (*a*) when the incident $\vec{E}$-field vector is vertical and (*b*) when the incident $\vec{E}$-field vector is horizontal. Random polarization is then an arbitrary linear combination of these two limiting cases:

(*a*.) $\vec{E}_{inc}$ vertical:



(*b*.) $\vec{E}_{inc}$ horizontal:



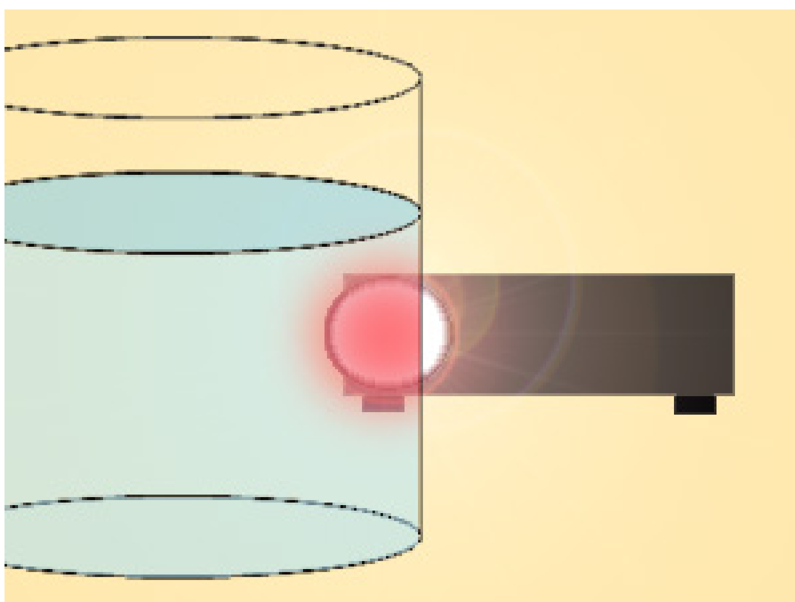
Because the blue light an observer sees from a given portion of the sky is due to the preferential scattering of E(1) electric dipole-type Rayleigh scattering of sunlight/solar EM radiation off of gas atoms in the Earth's atmosphere, with $\vec{E}_{scat} \perp$ to the line-of-sight, this radiation has a ***net*** polarization – *i.e.* the light from the sky is polarized, especially so away from the sun, *i.e.* in the northern portions of the sky {in the northern hemisphere} !!! You can very easily observe/explicitly verify this using a pair of polaroid sunglasses – try it some time!!!

It is beneficial to wear polaroid sunglasses *e.g.* when out boating on a lake – in order to reduce "glare" from {polarized} sunlight reflected off of the surface of the water!!!

As mentioned above, at sunrise or sunset, the sun appears red when an observer is looking directly at the sun, because the blue/violet light is ~ 25× more preferentially scattered out of the beam of light incident from the sun {per unit thickness of atmosphere} than red light. Thus sunlight when the sun is near the horizon consists predominantly of what remains – red light.

Note that this is also true for ***moonrise*** and ***moonset*** – the moon will {likewise} have a reddish hue at these times, and note that this is also true *e.g.* for the case of an eclipse of the moon by the Earth.

One can also observe this same phenomenon *e.g.* using a glass pitcher of milk diluted with water – because milk molecules are efficient Rayleigh scatterers of visible light! Here's a simple experiment that you can carry out at home, *e.g.* using a flashlight:



The {scalar} *EM* wave ***characteristic radiation impedance*** of an antenna is exactly as we defined the characteristic impedance of a waveguide; noting here that we are dealing with manifestly transverse waves for *EM* wave radiation from an E(1) electric dipole antenna:

$$\boxed{Z_{antenna}(\vec{r}) \equiv \frac{\left|\vec{E}_\perp^{rad}(\vec{r})\right|}{\left|\vec{H}_\perp^{rad}(\vec{r})\right|} = \frac{\left|\vec{E}_\perp^{rad}(\vec{r})\right|}{\frac{1}{\mu_o}\left|\vec{B}_\perp^{rad}(\vec{r})\right|} = \frac{\left|\vec{E}^{rad}(\vec{r})\right|}{\frac{1}{\mu_o}\left|\vec{B}^{rad}(\vec{r})\right|}}$$

Let's check the SI units of this definition:

$$\boxed{\left(\frac{E}{B/\mu_o}\right) = \frac{\left(Volts/m\right)}{\left(Teslas\Big/\frac{Henrys}{m}\right)} = \frac{\left(Volts/m\right)}{\left(N/A\text{-}m\right)/\left(N/A^2\right)} = \frac{Volts}{Amps} = Ohms}$$

For E(1) electric dipole radiation, the *EM* wave characteristic radiation impedance in the "far-zone" limit ($d \ll \lambda \ll r$) with $\boxed{c = 1/\sqrt{\varepsilon_o\mu_o}}$ is:

$$\boxed{\vec{E}_r^{E(1)}(\vec{r},t) \approx -\frac{\mu_o p\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\theta}} \text{ and: } \boxed{\vec{B}_r^{E(1)}(\vec{r},t) \approx -\frac{\mu_o p\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\left[\omega\left(t-\frac{r}{c}\right)\right]\hat{\varphi}}$$

$$\boxed{Z_{antenna}^{E(1)}(\vec{r}) = \frac{\mu_o \cdot \cancel{\frac{\mu_o p\omega^2}{4\pi}}\cancel{\left(\frac{\sin\theta}{r}\right)}\cancel{\cos\left[\omega\left(t-\frac{r}{c}\right)\right]}}{\cancel{\frac{\mu_o p\omega^2}{4\pi}}\frac{}{c}\cancel{\left(\frac{\sin\theta}{r}\right)}\cancel{\cos\left[\omega\left(t-\frac{r}{c}\right)\right]}} = \mu_o c = \sqrt{\frac{\mu_o}{\varepsilon_o}} \equiv Z_o = 120\pi\ \Omega = 377\ \Omega}$$

Where:

$\mu_o = 4\pi\times10^{-7}$ Henrys/m = magnetic permeability of free space / vacuum

$\varepsilon_o = 8.85\times10^{-12}$ Farads/m = electric permittivity of free space / vacuum

And: $$\boxed{Z_o \equiv \sqrt{\frac{\mu_o}{\varepsilon_o}} = \sqrt{\frac{4\pi\times10^{-7}\text{ Henrys/m}}{8.85\times10^{-12}\text{ Farads/m}}} = 120\pi\ \Omega = 377\ \Omega}$$ = {scalar} characteristic impedance of free space/the vacuum.

Thus we see that E(1) electric dipole antennae (in the "far-zone" limit ($d = \pi b \ll \lambda \ll r$)) are ***perfectly* impedance-matched** for propagation of E(1) *EM* waves into free space / vacuum!

Note also that the "far-zone" ($d \ll \lambda \ll r$) *EM* wave characteristic radiation impedance $Z_{antenna}^{E(1)}(\vec{r})$ has ***no*** spatial and/or frequency dependence.

The *EM* Wave Radiation Resistance of an Antenna:

The {scalar} *EM* wave ***radiation resistance*** of an antenna R_{rad} is defined in terms of the antenna power P_{rad} and the amplitude of the current I flowing in the antenna:

$$\boxed{P_{rad}^{antenna} \equiv I^2 R_{rad}^{antenna}} \quad \text{or:} \quad \boxed{R_{rad}^{antenna} \equiv P_{rad}^{antenna} / I^2} \quad \text{(Ohms)}$$

For an E(1) electric dipole antenna: $I = q\omega$ = amplitude of current flowing in the dipole.

In the "far-zone" limit, *i.e.* $d \ll \lambda \ll r$:

$$\boxed{R_{rad}^{\mathrm{E}(1)} \approx \frac{\mu_o p^2 \omega^4}{12\pi c^3 I^2} = \frac{\mu_o \cancel{q^2} d^2 \omega^{\cancel{4}}}{12\pi c \cancel{q^2} \cancel{\omega^2}} = \frac{\mu_o \omega^2 d^2}{12\pi c} (\Omega)}$$

n.b. $R_{rad}^{\mathrm{E}(1)}$ is frequency-dependent!

In the "far-zone" limit, *i.e.* $d \ll \lambda \ll r$:

$$\boxed{R_{rad}^{\mathrm{E}(1)} \approx \frac{\mu_o \omega^2 d^2}{12\pi c} = \frac{\omega^2 d^2}{12\pi c^2}(\mu_o c) = \frac{\omega^2 d^2}{12\pi c^2}\sqrt{\frac{\mu_o}{\varepsilon_o}} = \frac{\omega^2 d^2}{12\pi c^2} Z_{rad}^{\mathrm{E}(1)}} \quad \text{But: } \boxed{Z_o = \sqrt{\frac{\mu_o}{\varepsilon_o}} = Z_{rad}^{\mathrm{E}(1)}}$$

$\therefore$ In the "far-zone" limit, $d \ll \lambda \ll r$:

$$\boxed{R_{rad}^{\mathrm{E}(1)} \approx \frac{\omega^2 d^2}{12\pi c^2} Z_o = \frac{1}{12\pi}\left(\frac{\omega d}{c}\right)^2 Z_o}$$

However, in the "far-zone" limit, $d \ll \lambda \ll r$ we have: $\boxed{\left(\frac{\omega d}{c}\right) \ll 1}$

Thus, we see that the *EM* wave ***radiation resistance*** $R_{rad}^{\mathrm{E}(1)}$ associated with E(1) electric dipole antenna in the "far-zone" limit ($d \ll \lambda \ll r$) is ***much*** less than the *EM* wave ***characteristic radiation impedance*** $\boxed{Z_{rad}^{\mathrm{E}(1),\mathrm{M}(1)} = Z_o = 120\pi\ \Omega \simeq 377\ \Omega}$ of an electric dipole antenna:

$$\boxed{R_{rad}^{\mathrm{E}(1)} \approx \frac{1}{12\pi}\left(\frac{\omega d}{c}\right)^2 Z_o \ll Z_o = 377\,\Omega}$$

