Energy & Momentum In Electromagnetic Waves

In a region of empty space where $\vec{E}$ and $\vec{B}$ fields are present, the total energy density u is given by:

$$u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2$$

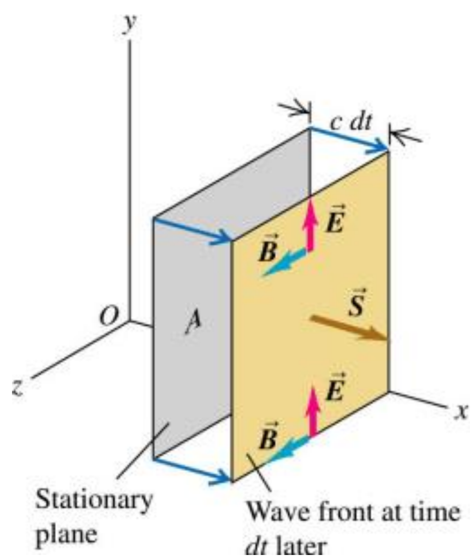
For electromagnetic waves in a vacuum,

$$B = \frac{E}{c}$$
$$= \sqrt{\epsilon_0 \mu_0}E$$

It follows that:

$$u = \epsilon_0 E^2$$

In a vacuum, the energy density associated with the **E** field is equal to the energy of the **B** field. In general, the energy density u of an electromagnetic wave depends on position and time. Electromagnetic waves transport energy from one region to another – they carry the energy density u with them as they advance.



Consider a stationary plane, perpendicular to the x-axis, that coincides with the wave front at a certain time. In a time dt after this, the wave front moves a distance dx=c dt to the right of the plane.
Consider an area A on this stationary plane, the energy dU in the space to the right of this area must have passed through the area to reach the new location.
Hence,

$$\begin{aligned} dU &= u\,dV \\ &= \left(\epsilon_0 E^2\right)(Ac\,dt) \end{aligned}$$

The energy flow per unit time per unit area is given by:

$$\begin{aligned} S &= \frac{1}{A}\frac{dU}{dt} \\ &= \epsilon_0 c E^2 \\ &= \epsilon_0 c^2 E\left(\frac{1}{c}E\right) \\ &= \frac{1}{\mu_0}EB \end{aligned}$$

The energy flow per unit time per unit area has a term attached to it: **Poynting vector**, **S**, where the direction is in the direction of propagation of the wave.

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$
$$S = \frac{1}{\mu_0}EB$$

The total energy flow per unit time out of any closed surface is given by:

$$P = \oint \vec{S} . d\vec{A}$$

Let's calculate the Poynting vector for typical sinusoidal waves:

$$\vec{E}(x,t) = E_{\text{max}} \cos(kx - \omega t)\hat{j}$$
$$\vec{B}(x,t) = B_{\text{max}} \cos(kx - \omega t)\hat{k}$$

$$\begin{aligned}\vec{S}(x,t) &= \frac{1}{\mu_0}\vec{E}(x,t) \times \vec{B}(x,t) \\ &= \frac{1}{\mu_0} E_{\text{max}} B_{\text{max}} \cos^2(kx - \omega t)\hat{i} \\ &= \frac{1}{2\mu_0} E_{\text{max}} B_{\text{max}} [1 + \cos 2(kx - \omega t)]\, \hat{i}\end{aligned}$$

The **intensity** of the radiation is the magnitude of the average value of the Poynting vector,

$$\begin{aligned}I &= S_{\text{average}} \\ &= \frac{E_{\text{max}} B_{\text{max}}}{2\mu_0} \\ &= \frac{1}{2\mu_0 c} E_{\text{max}}^2 \\ &= \frac{c}{2\mu_0 c^2} E_{\text{max}}^2 \\ &= \frac{1}{2}\epsilon_0 c E_{\text{max}}^2 \\ &= \frac{1}{2}\sqrt{\frac{\epsilon_0}{\mu_0}} E_{\text{max}}^2\end{aligned}$$

Electromagnetic waves also carry momentum pp, with a corresponding momentum density. Let's calculate the momentum carried by electromagnetic waves by using the well known relativistic formula:

$$E^2 = p^2c^2 + m^2c^4$$

According to quantum mechanics, the electromagnetic radiation is made up of massless particles called photons, with momentum $p = E/c$ for individual photons. It follows from $p = E/c$ that the **momentum density** for electromagnetic waves must be equal to the energy density divided by c. Since the energy density for electromagnetic waves is given by:

$$\begin{aligned}\frac{dp}{dV} &= \frac{\epsilon_0 E^2}{c} \\ &= \epsilon_0 EB \\ &= \frac{EB}{\mu_0 c^2} \\ &= \frac{S}{c^2}\end{aligned}$$

We can further express the above as **momentum transferred per unit time per unit area**:

$$\begin{aligned}\frac{dp}{dV} &= \frac{S}{c^2} \\ \frac{dp}{Ac\,dt} &= \frac{S}{c^2} \\ \frac{1}{A}\frac{dp}{dt} &= \frac{S}{c} \\ \frac{1}{A}\frac{dp}{dt} &= \frac{EB}{\mu_0 c}\end{aligned}$$

This momentum is a property of the field – it is not associated with the mass of a moving particle in the usual sense. This momentum is responsible for the phenomenon of radiation pressure. If an electromagnetic wave with an average value of Poynting vector of S_{av} is incident on an object, with no reflection and transmission, the **radiation pressure** on the object will be given by: (NOTE: p_{rad} is radiation pressure and dp is the infinitesimal change in momentum.)

$$\begin{aligned} p_{\text{rad}} &= \frac{F_{\text{rad}}}{A} \\ &= \frac{1}{A}\frac{dp}{dt} \\ &= \frac{S_{\text{av}}}{c} \\ &= \frac{I}{c} \end{aligned}$$

If all of the incident electromagnetic waves are reflected by the object, the resulting radiation pressure will be:

$$\begin{aligned} p_{\text{rad}} &= \frac{2S_{\text{av}}}{c} \\ &= \frac{2I}{c} \end{aligned}$$

Interaction of Atoms and Electromagnetic Waves

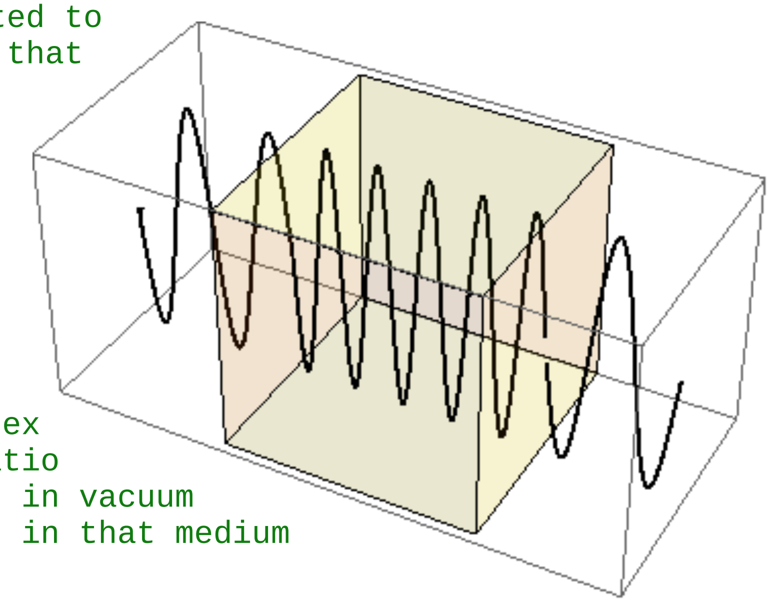
<u>Outline</u>

- Review: Polarization and Dipoles
- Lorentz Oscillator Model of an Atom
- Dielectric constant and Refractive index

Refractive Index: Waves in Materials

Speed of light in any medium is related to the permittivity and permeability in that medium

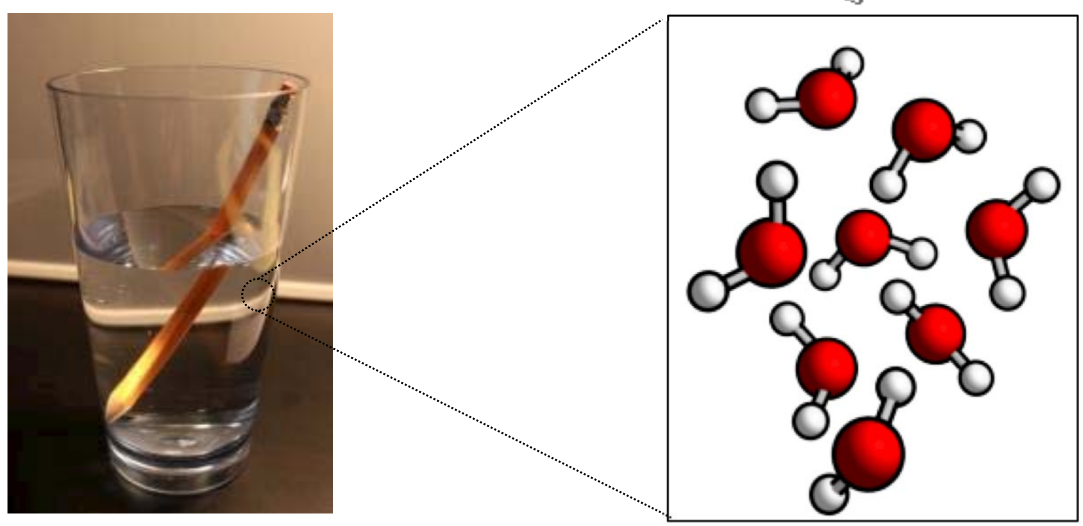
$$v_p = \frac{1}{\sqrt{\mu\epsilon}}$$



Index of refraction

$$n \equiv \frac{c}{v_p}$$

Definition of index of refraction: ratio of speed of light in vacuum to speed of light in that medium



How do we get from molecules/charges and fields to index of refraction ?

Index of Refraction

$$\nu\lambda = v_p = c/n$$

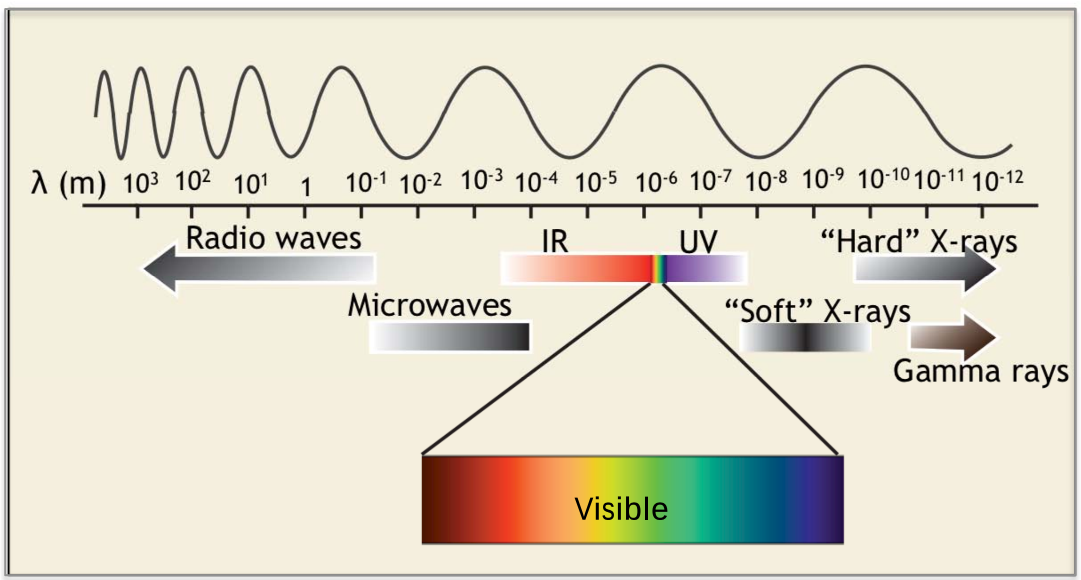
frequency   wavelength

When propagating in a material,

$$c \to c/n$$
$$\lambda \to \lambda_o/n$$
$$k \to k_o/n$$

Should be * not /

Material	n
Vacuum	1
Air	1.000277
Water liquid	1.3330
Water ice	1.31
Diamond	2.419
Silicon	3.96
at 5 x 10^{14} Hz	



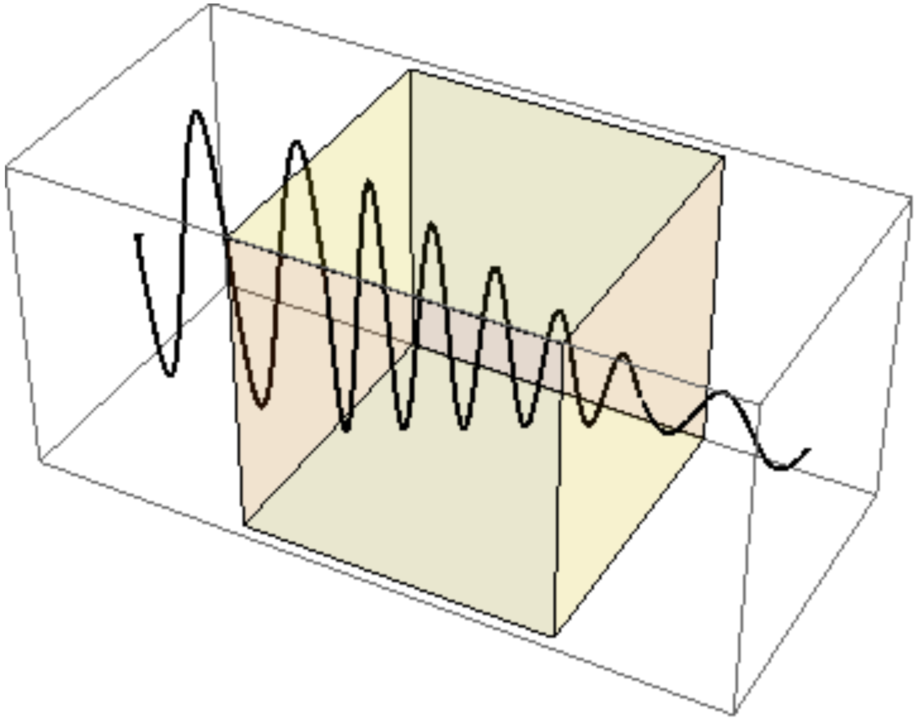
Propagating EM wave

$$E(t,z) = Re\{\tilde{E}_0 e^{j(\omega t - k_0 n z)}\}$$

$$E(t,z) = Re\{\tilde{E}_0 e^{j(\omega t - k\ z)}\}$$

Absorption

Photograph by Hey Paul on Flickr.



Why are these stained glass different colors?

$$E(t, z) = Re\{\tilde{E}_0 e^{-\alpha z/2} e^{j(\omega t - k_0 n z)}\}$$

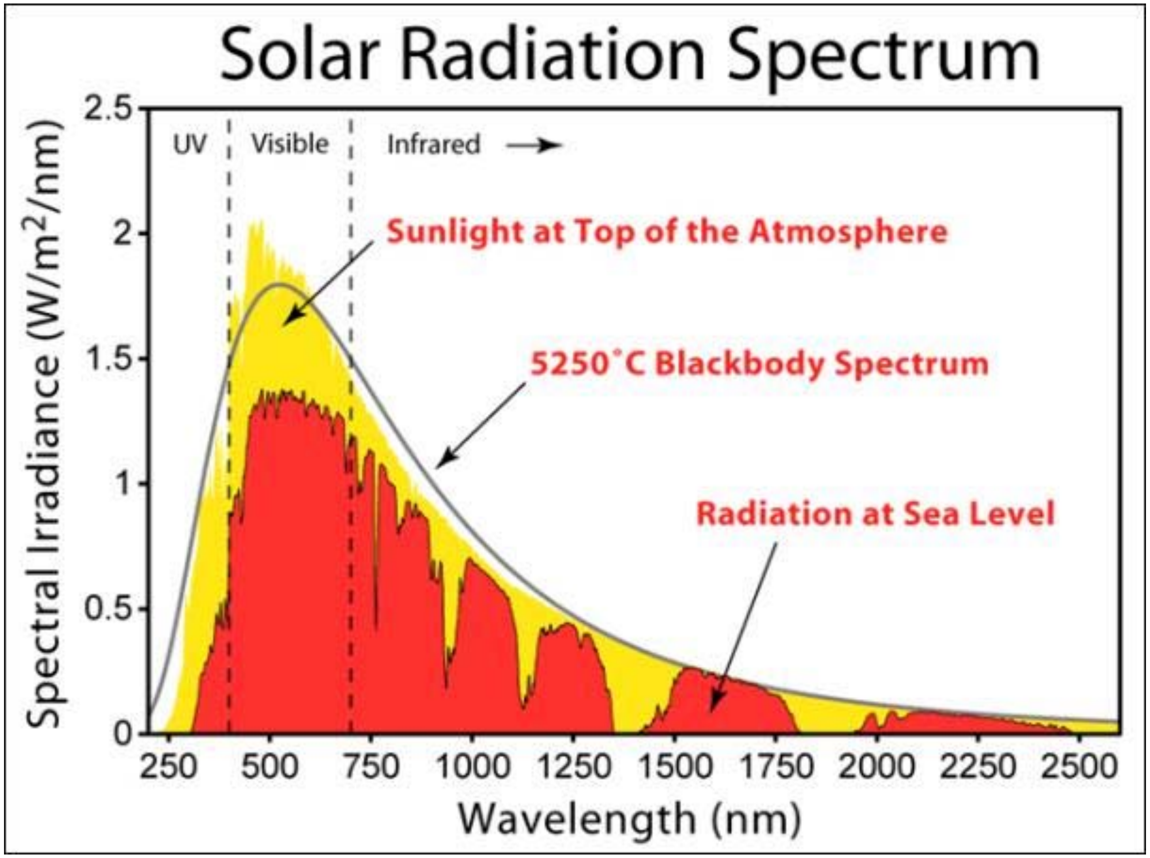
Absorption coefficient (α)

Refractive index (n)

Tomorrow: lump refractive index and absorption into a complex refractive index $\tilde{n}$

Assume there is some attenuation in exponential form. Now introduce the concept of a complex refractive index to merge effects.

Incident Solar Radiation



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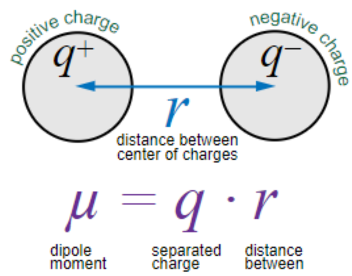
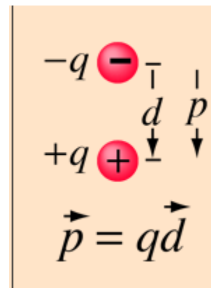
How do we introduce propagation through a medium (atmosphere) into Maxwell's Equations?

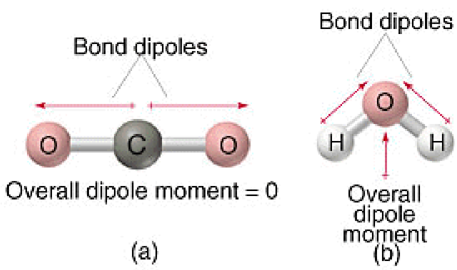
Dipole Moment

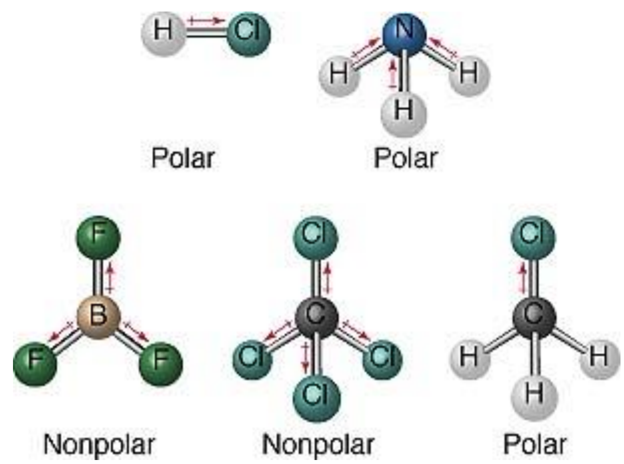
A dipole moment is a quantity that describes two opposite charges separated by a distance. It is a quantity that we can measure for a molecule in the lab and thereby determine the size of the partial charges on the molecule (if we know the bond length). By definition the dipole moment, μ, is the product of the magnitude of the separated charge and the distance of the separation:





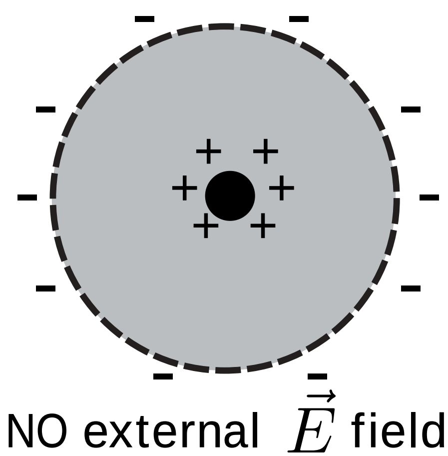
where q is the magnitude of the separated charge and r is the distance between them.

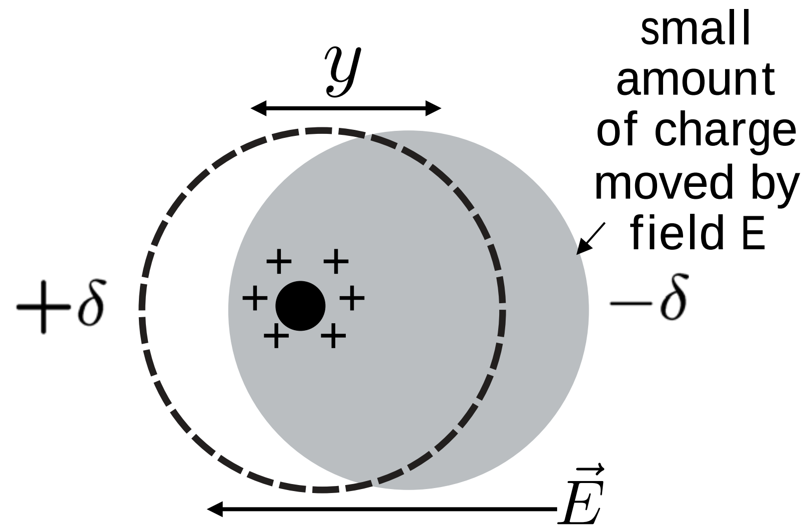


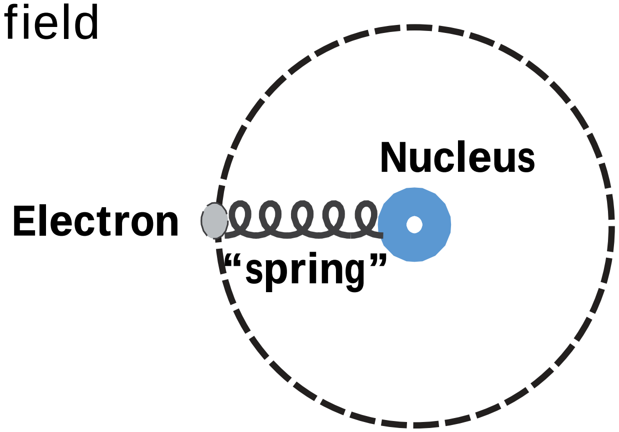


Compounds	Dipole Moment	Structure
HCl	1.03D	δ+ δ− H⟶Cl
HBr	0.79D	δ+ δ− H⟶Br
HI	0.38D	δ+ δ− H⟶I
HF	2.00D	δ+ δ− H⟶F

Microscopic Description of Dielectric Constant







Density of dipoles...

$$\vec{P} = N\delta\vec{x}$$

... equivalent to ...

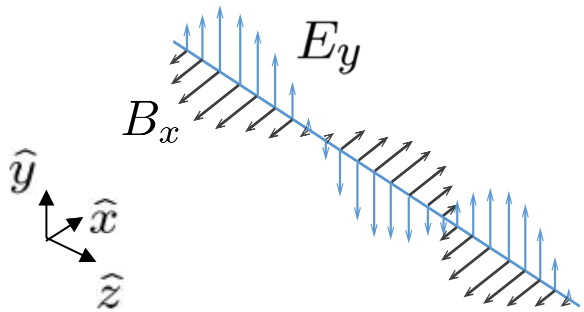
Electric field polarizes molecules...

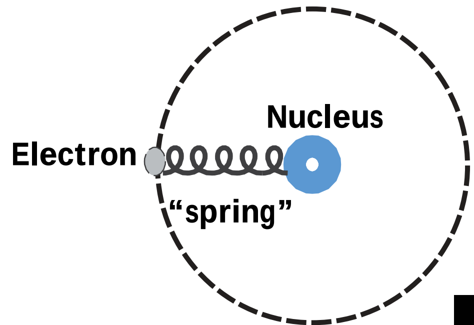
$$\vec{P} = \epsilon_o \chi_e \vec{E}$$

X is susceptibility and relates permittivity in a medium to that in vacuum.

Lorentz Oscillator

Lorentz was a late nineteenth century physicist, and quantum mechanics had not yet been discovered. However, he did understand the results of classical mechanics and electromagnetic theory. Therefore, he described the problem of atom-field interactions in these terms. Lorentz thought of an atom as a mass (the nucleus) connected to another smaller mass (the electron) by a spring. The spring would be set into motion by an electric field interacting with the charge of the electron. The field would either repel or attract the electron which would result in either compressing or stretching the spring.





Hendrik Lorentz (1853-1928)
Nobel in 1902 for Zeeman Effect

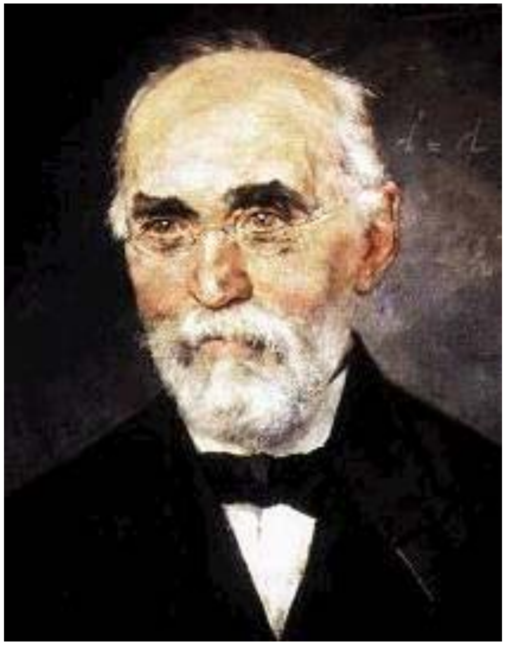


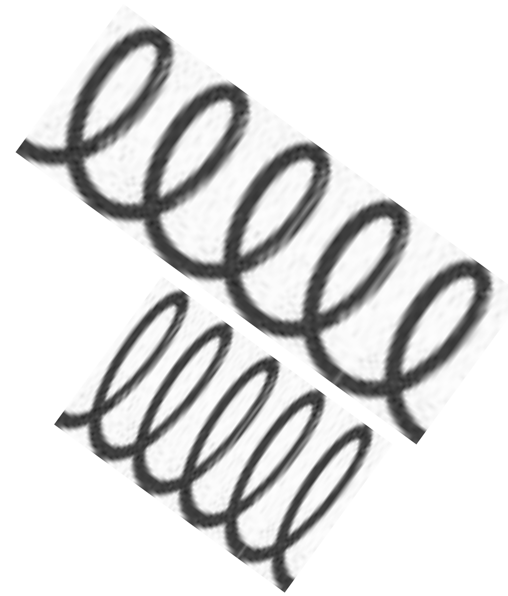
Image in public domain

Lorentz was not positing the existence of a physical spring connecting the electron to an atom; however, he did postulate that the force binding the two could be described by Hooke's Law:

$$F(y) = -k_s y$$

where y is the displacement from equilibrium. If Lorentz's system comes into contact with an electric field, then the electron will simply be displaced from equilibrium. The oscillating electric field of the electromagnetic wave will set the electron into harmonic motion. The effect of the magnetic field can be omitted because it is miniscule compared to the electric field.

Springs have a resonant frequency



Hooke's Law $$m\frac{d^2y}{dt^2} = -ky$$

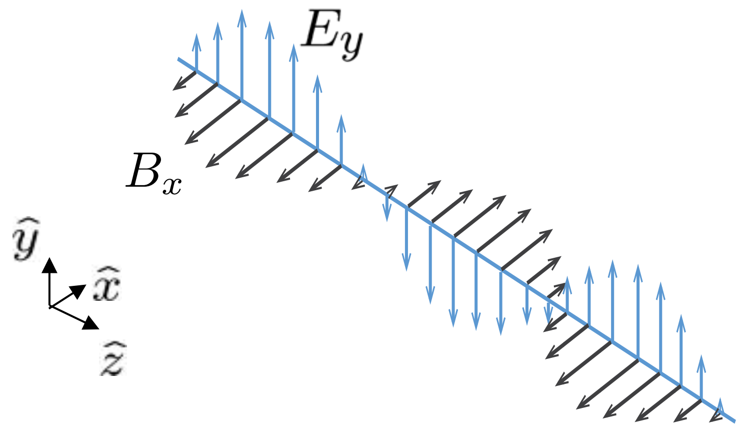
$$\frac{d^2y}{dt^2} = -\frac{k}{m}y$$

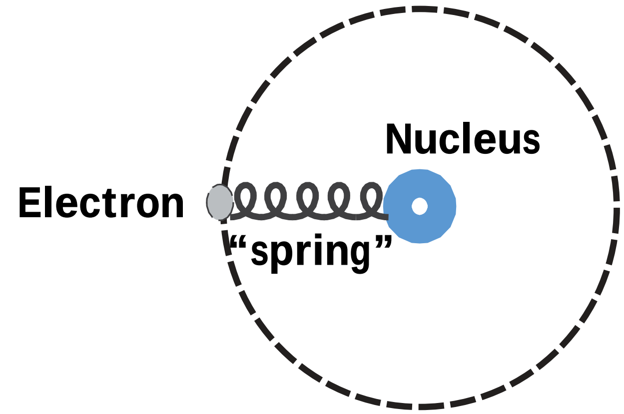
Solution $$y(t) = Asin(t\underbrace{\sqrt{\frac{k}{m}}}_{\omega_0}) + Bcos(t\underbrace{\sqrt{\frac{k}{m}}}_{\omega_0})$$

So we can write: $$\omega_o^2 = \frac{k}{m} \quad \Rightarrow \quad k = m\omega_o^2$$

Using classical spring solution, the spring constant is related to a natural frequency. No more "k" terms are used.

Microscopic Description of Dielectric Constant





Replace "k", add coulomb force, and assume that there is some classic damping proportional to velocity

$$m\frac{d^2y}{dt^2} = -m\omega_o^2 y + qE_y - m\gamma\frac{dy}{dt}$$

Electron mass

Restoring force (binding electron & nucleus)

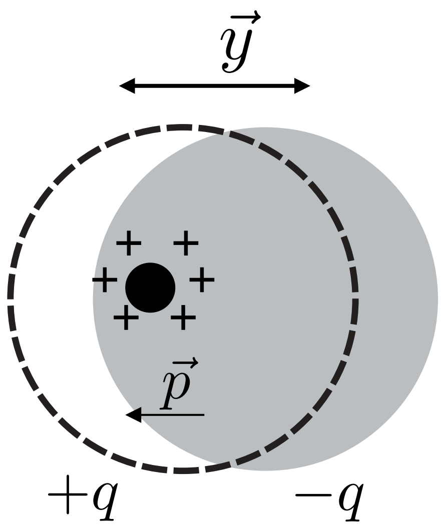
$\vec{E}$ field force

Damping

Solution using complex variables

Lets plug-in the expressions for E_y and y
into the differential equation from slide 9:

Natural resonance



$$\frac{d^2}{dt^2}y(t) + \gamma\frac{d}{dt}y(t) + \omega_o^2 y(t) = \frac{q}{m}E_y(t)$$

$$E_y(t) = Re\{E_y e^{j\omega t}\} \qquad y(t) = Re\{y e^{j\omega t}\}$$

$$\omega^2 y + j\omega\gamma y + \omega_o^2 y = \frac{q}{m}E_y$$

$$y = \frac{q}{m}\frac{1}{(\omega_o^2 - \omega^2) + j\omega\gamma}E_y$$

This is a lot like the classic mass-spring solution.

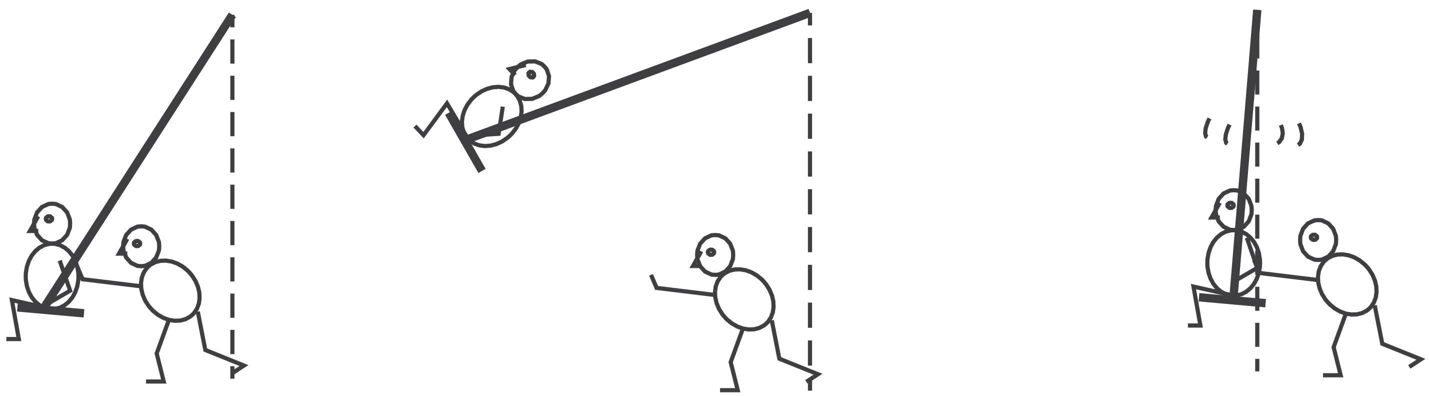
Oscillator Resonance

$$y = \frac{q}{m} \frac{1}{(\omega_0^2 - \omega^2) + j\omega\gamma} E_y$$

$$E_y(t) = Re\{E_y e^{j\omega t}\}$$

$$y(t) = Re\{y e^{j\omega t}\}$$

Driven harmonic oscillator: **Amplitude** and **Phase** depend on frequency



Low frequency

medium amplitude

Displacement, y
in phase with E_y

At resonance

large amplitude

Displacement, y
90º out of phase with E_y

High frequency

vanishing amplitude

Displacement y and E_y
in antiphase

Polarization

Since charge displacement, y, is directly related to polarization, P, of our material we can then rewrite the differential equation:

$$\vec{D} = \epsilon_o \vec{E} + \vec{P}$$

$$P_y = Nqy$$

For linear polarization in $\hat{y}$ direction

$$\left(\frac{d}{dt^2} + \gamma\frac{d}{dt} + \omega_o^2\right)P_y(t) = \frac{Nq^2}{m}E_y(t) = \epsilon_o\omega_p^2 E_y(t)$$

$$\omega_p^2 = \frac{Nq^2}{\epsilon_o m}$$

This mostly follows directly from the way P and D are defined.

$$P_y(t) = Re\{P_y e^{j\omega t}\}$$

By defnintion, term in red is just susceptibility or permittivity over permittivity in vacuum.

$$P_y = \frac{\omega_p^2}{(\omega_o^2 - \omega^2) + j\gamma\omega}\epsilon_o E_y$$

Dielectric Constant from the Lorentz Model

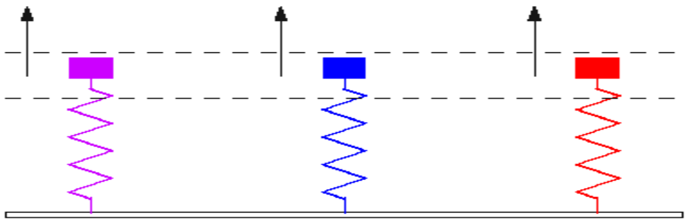
$$\vec{D} = \epsilon_o \vec{E} + \vec{P}$$

$$\vec{P} = \frac{\omega_p^2}{(\omega_o^2 - \omega^2) + j\gamma\omega} \epsilon_o \vec{E}$$

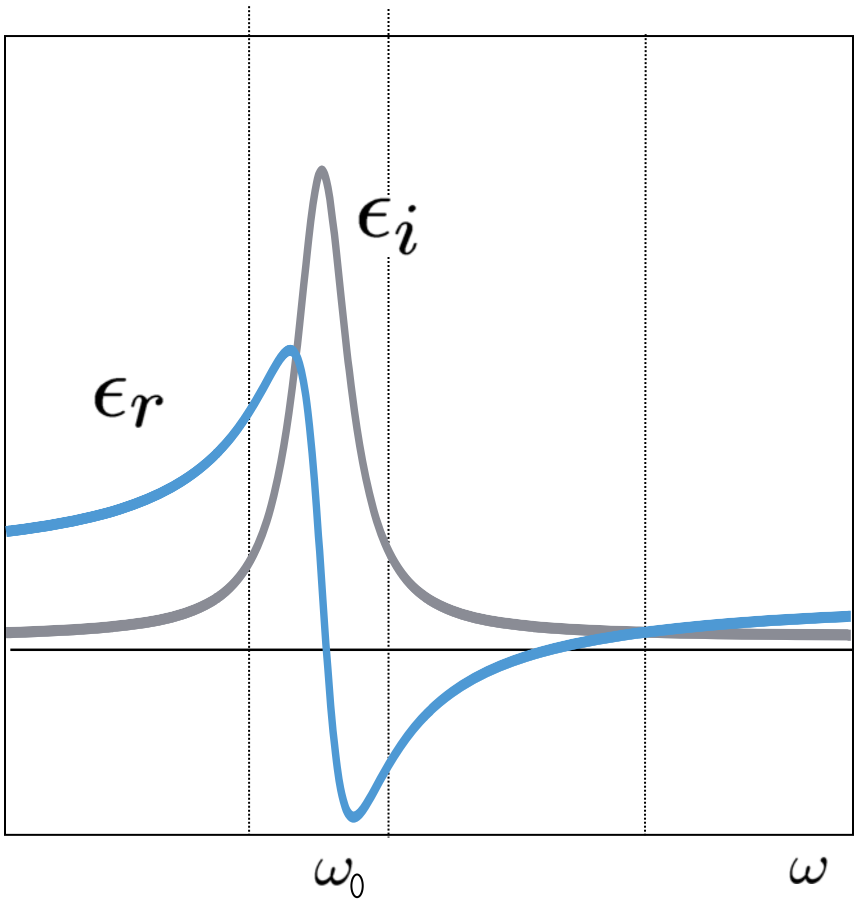
$$\vec{D} = \epsilon_o \left[1 + \frac{\omega_p^2}{(\omega_o^2 - \omega^2) + j\gamma\omega} \right] \vec{E}$$

$$\boxed{\epsilon = \epsilon_o \left[1 + \frac{\omega_p^2}{(\omega_o^2 - \omega^2) + j\gamma\omega} \right]}$$

Microscopic Lorentz Oscillator Model



$$\vec{P} = \frac{\omega_p^2}{(\omega_o^2 - \omega^2) + j\gamma\omega}\epsilon_o\vec{E}$$



$$\omega_p^2 = \frac{Nq^2}{\epsilon_o m}$$

$$\epsilon = \epsilon_r - j\epsilon_i$$

Key physics is done at this point. The rest is definitions. Breaking down the term in front of E to real and imaginary components is first step.

Real and imaginary parts

$$\epsilon = \frac{\omega_p^2}{(\omega_o^2 - \omega^2) + j\omega\gamma}$$

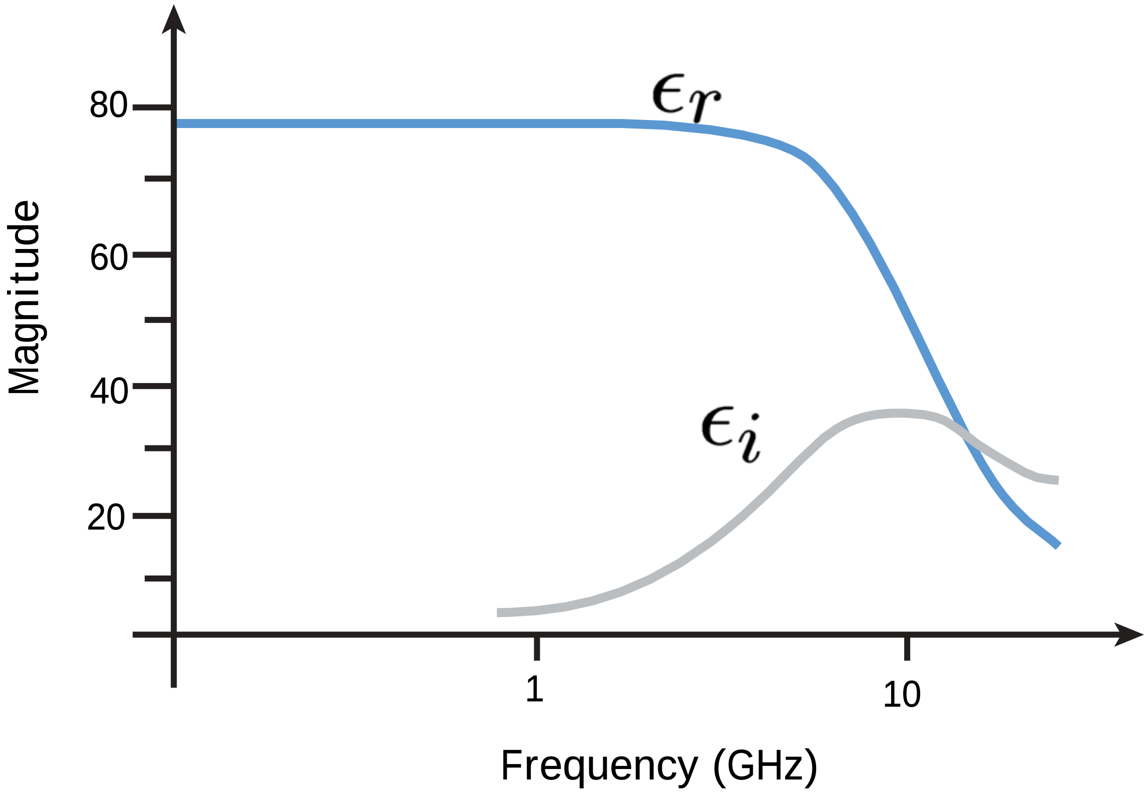
$$= \frac{\omega_p^2(\omega_o^2 - \omega^2)}{(\omega_o^2 - \omega^2)^2 + \omega^2\gamma^2} - j\frac{\omega_p^2\omega\gamma}{(\omega_o^2 - \omega^2)^2 + \omega^2\gamma^2}$$

$$\epsilon = \epsilon_r - j\epsilon_i$$

$$\epsilon_r = \frac{\omega_p^2(\omega_o^2 - \omega^2)}{(\omega_o^2 - \omega^2)^2 + \omega^2\gamma^2} \qquad \epsilon_i = \frac{\omega_p^2\omega\gamma}{(\omega_o^2 - \omega^2)^2 + \omega^2\gamma^2}$$

Real part has the difference term in numerator and denominator, so it won't peak on resonance.
But imaginary part clearly does peak on resonance.

Dielectric constant of water



Microwave ovens usually operate at 2.45 GHz

Not sure what the author is saying here. Maybe that microwaves operate at a wavelength where both e_i and e_r are high?

Recall that speed of light c in vacuum is $1/\sqrt{\mu_o \epsilon_o}$, and speed of light in any medium c_m is similarly $1/\sqrt{\mu\epsilon}$. If we define index of refraction in terms of ratio of speeds of light then:

$n = \left(\frac{\mu\epsilon}{\mu_o\epsilon_o}\right)^{1/2}$ and if the medium is non-magnetic such that $\mu=\mu_o$, then $n = \left(\frac{\mu_o\epsilon}{\mu_o\epsilon_o}\right)^{1/2}$ and

$\epsilon/\epsilon_o = n^2$

As defined in the slides, when they define their ε, it's really $\varepsilon/\varepsilon_o$ since the ω_p^2 term has a $1/\varepsilon_o$ term in it. So that's how they can say $\varepsilon = n^2$.

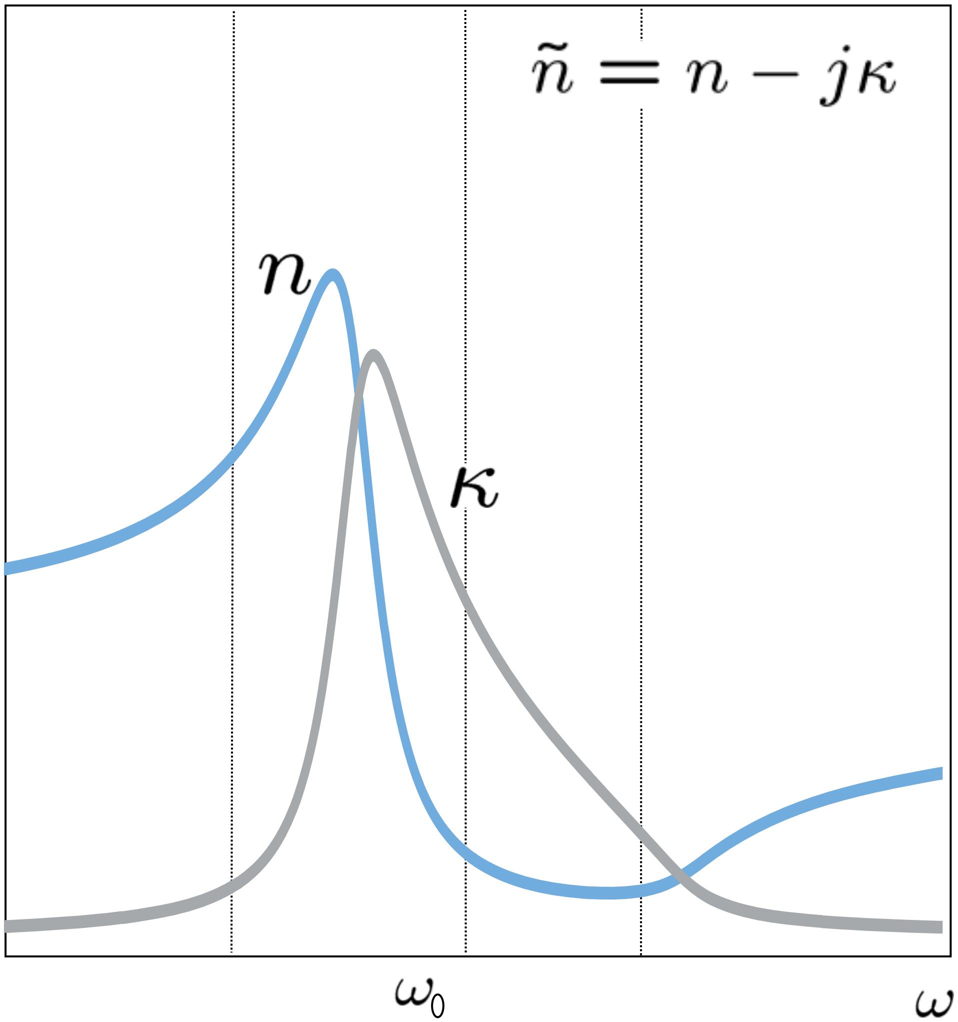
But if ε is complex, then it's reasonable to define an *n* that is complex with real part *n* and imaginary part κ. Then you can relate n and κ to ε_i and ε_r.

When you do all this and relate to the proposed form of the electric field:

$$E(t, z) = Re\{\tilde{E}_0 e^{-\alpha z/2} e^{j(\omega t - k_0 n z)}\}$$

You find that a is directly proportional to κ. Thus, you often hear, "The absorption coefficient is the imaginary part of the index of refraction."

Complex Refractive Index



$$\tilde{n} = n - j\kappa$$

$$\tilde{\epsilon} = \tilde{n}^2$$

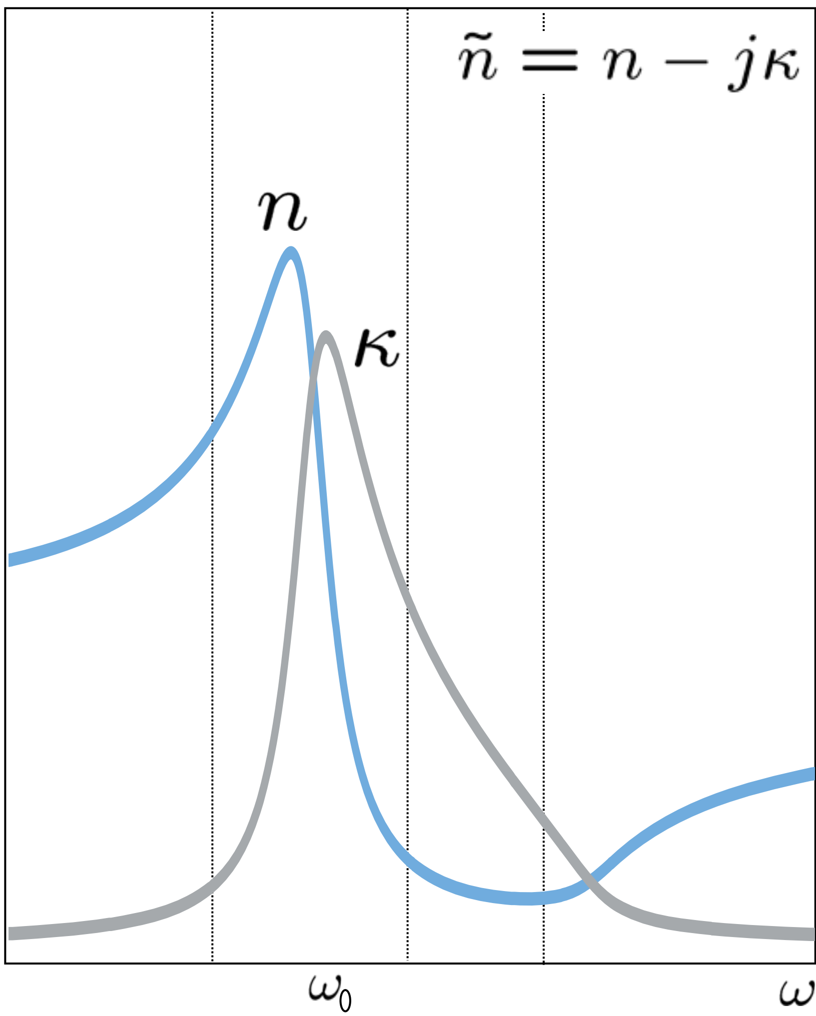
$$= (n - j\kappa)^2$$

$$= n^2 - \kappa^2 - 2jn\kappa$$

$$\epsilon_r = n^2 - \kappa^2$$

$$\epsilon_i = 2n_r n_i$$

Absorption Coefficient



$$n = \frac{1}{\sqrt{2}}\sqrt{\epsilon_r + \sqrt{\epsilon_r^2 + \epsilon_i^2}}$$

$$\kappa = \frac{1}{\sqrt{2}}\sqrt{-\epsilon_r + \sqrt{\epsilon_r^2 + \epsilon_i^2}}$$

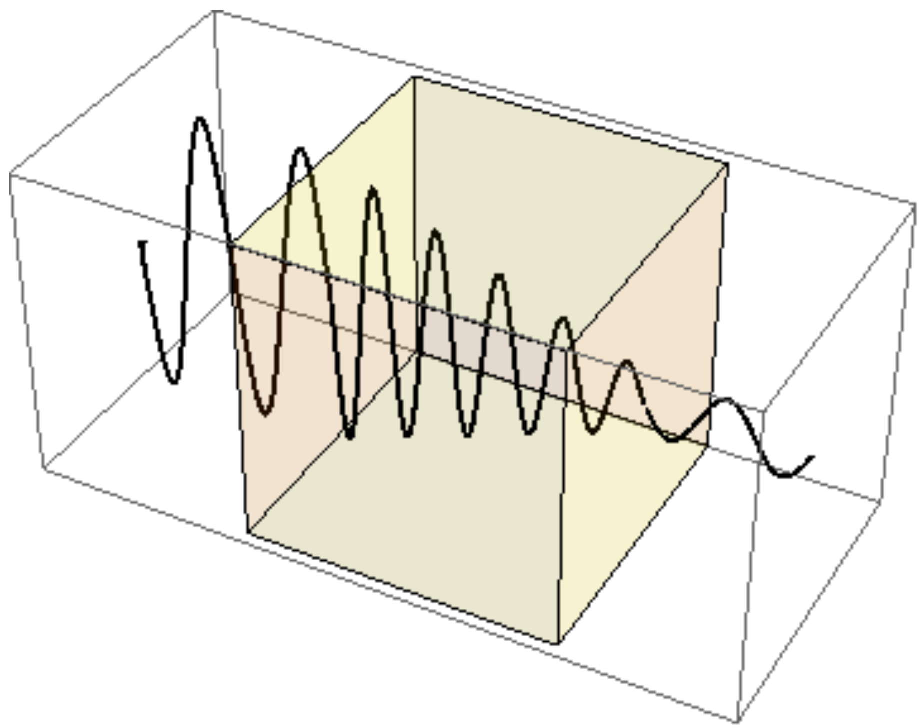
$$E(t,z) = Re\{\tilde{E}_0 e^{-\alpha z/2} e^{j(\omega t - k_0 n z)}\}$$

Absorption

Refractive index

$$\alpha = 2k_o\kappa = 2\frac{2\pi}{\lambda_o}\kappa \quad [\text{cm}^{-1}]$$

Absorption



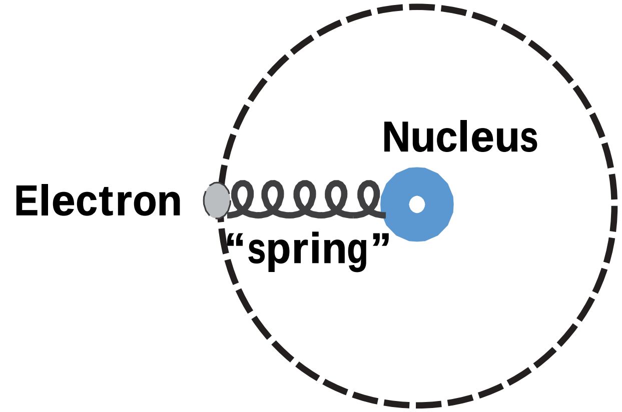
$$E(t,z) = Re\{\tilde{E}_0 e^{-\alpha z/2} e^{j(\omega t - k_0 n z)}\}$$

Absorption coefficient

Refractive index

$$I(z) = I_o e^{-\alpha z}$$ Beer-Lambert Law or Beer’s Law

Key Takeaways



Lorentz Oscillator Model

$$m\frac{d^2y}{dt^2} = -m\omega_o^2 y + qE_y - m\gamma\frac{dy}{dt}$$

$$\vec{P} = \frac{\omega_p^2}{(\omega_o^2 - \omega^2) + j\gamma\omega}\epsilon_o \vec{E}$$

$\Rightarrow$ $\tilde{n} = n - j\kappa$

$\Rightarrow$ $\epsilon = \epsilon_r - j\epsilon_i$

$$\alpha = 2k_o\kappa$$

Decay

$$E(t,z) = Re\{\tilde{E}_0 e^{-\alpha z/2} e^{j(\omega t - k_0 n z)}\}$$

Absorption coefficient

Refractive index

$$I(z) = I_o e^{-\alpha z}$$ Beer’s Law