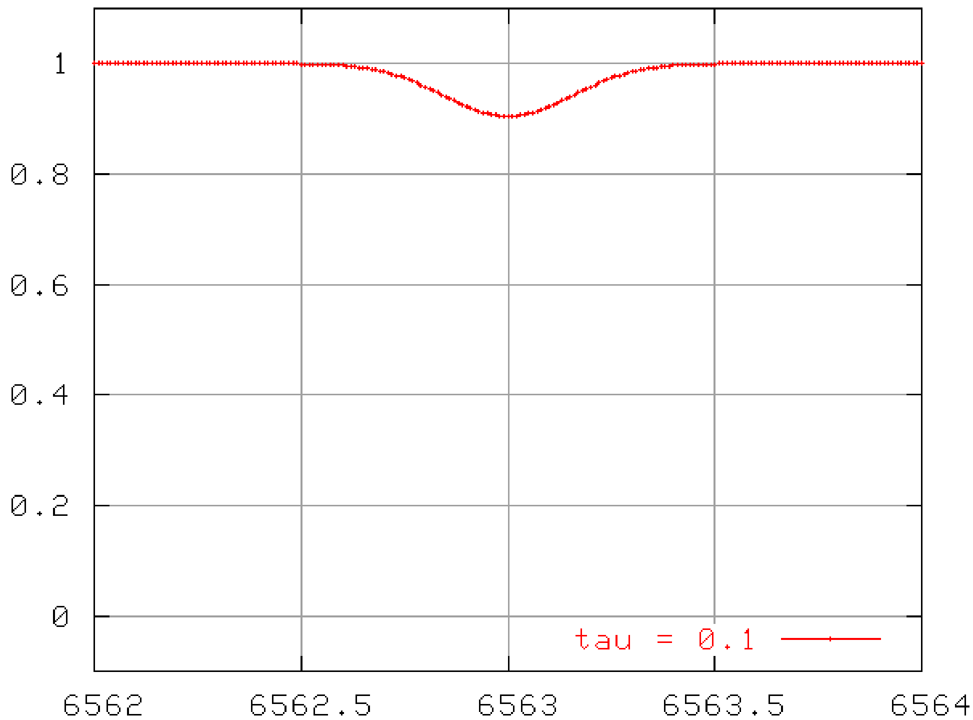
The curve of growth

From Michael Richmond, RIT, Phys440

So, we measure a stellar spectrum, and notice a very weak absorption line. This happens to correspond to an absorption line which has an optical depth $\tau = 0.1$. That means that most of the photons of the central wavelength (6563 Angstroms in the example below) produced within the photosphere manage to escape before being scattered or absorbed.



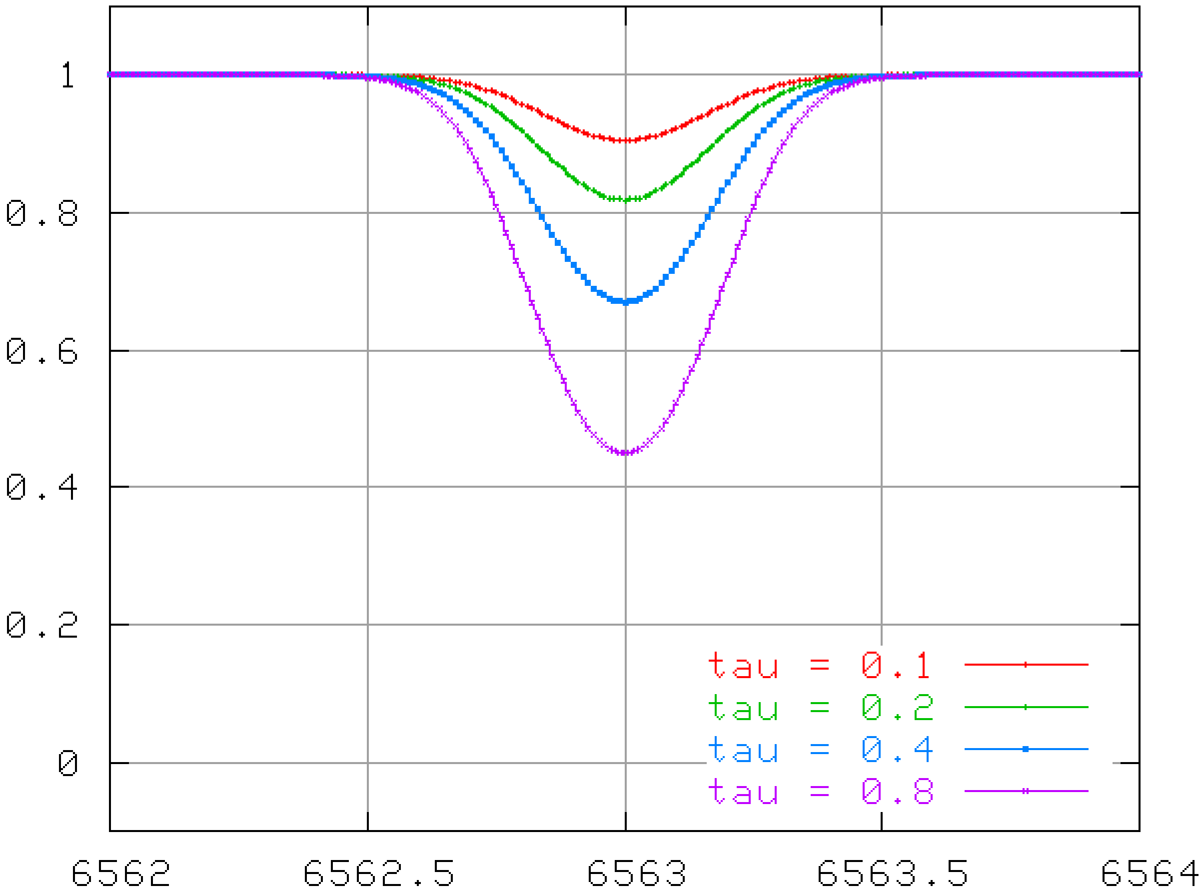
Thermally broadened lines

For ordinary stars like the Sun, weak lines are dominated by Doppler broadening. The width of the line is caused by random motions of the atoms absorbing the light. How does Doppler broadening grow as we increase the number of absorbing atoms?

Recall that the optical depth can be defined as

$$\frac{I}{I_0} = e^{-\kappa \rho s}$$
$$= e^{-\tau}$$

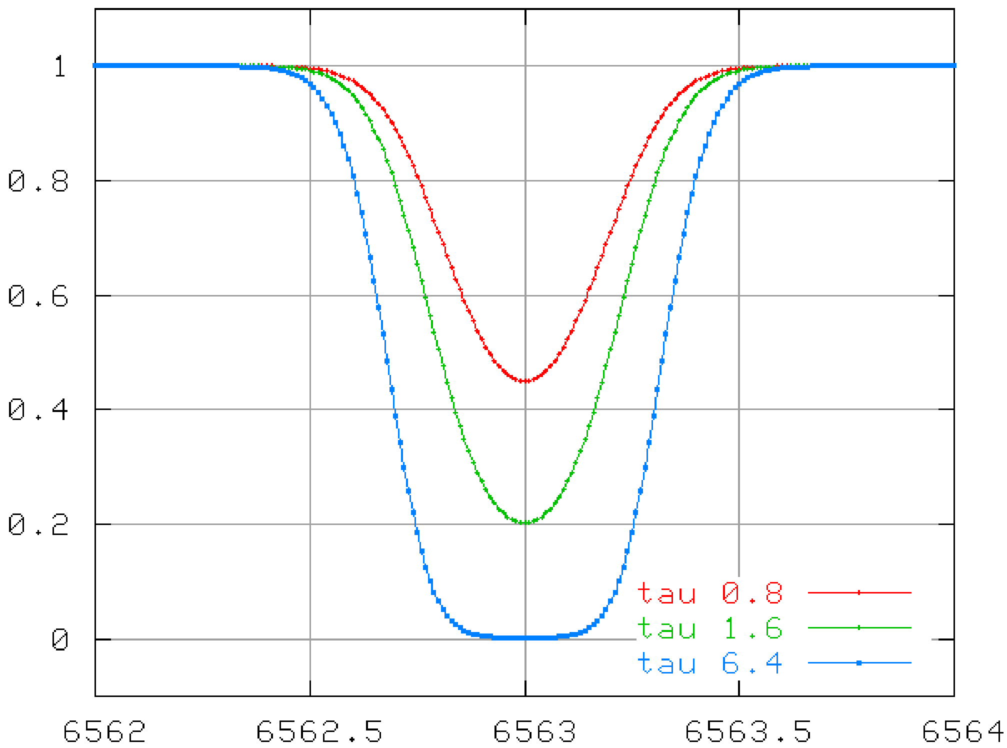
So, if we double the density **ρ** (and hence number of atoms), we double the optical depth. How does this affect the equivalent width? For thermal broadening, one can calculate the change in the absorption profile and find ....



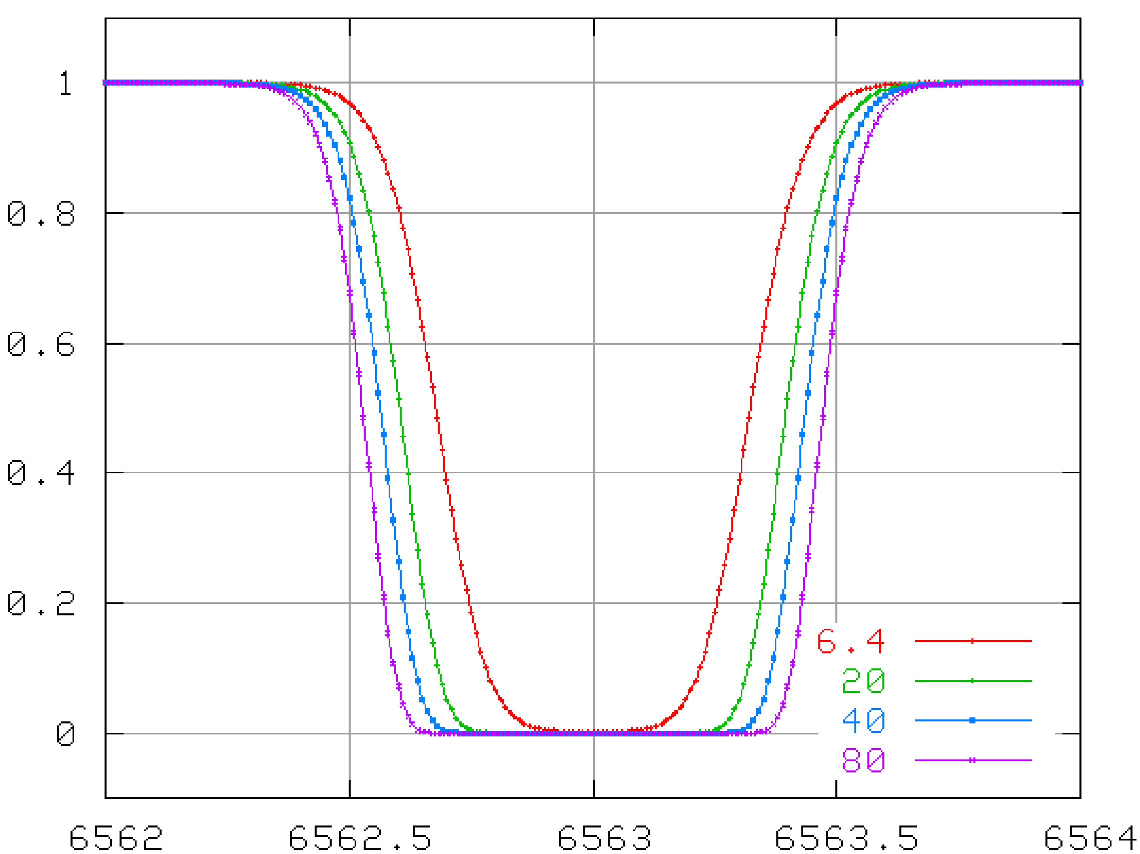
At first, this doubling of the optical depth causes a doubling of the equivalent width of the line

- There is a **linear** relationship between number of atoms and equivalent width for an optically thin line.

But as the line becomes optically thick, the equivalent width no longer grows in step with the optical depth (and number of atoms):

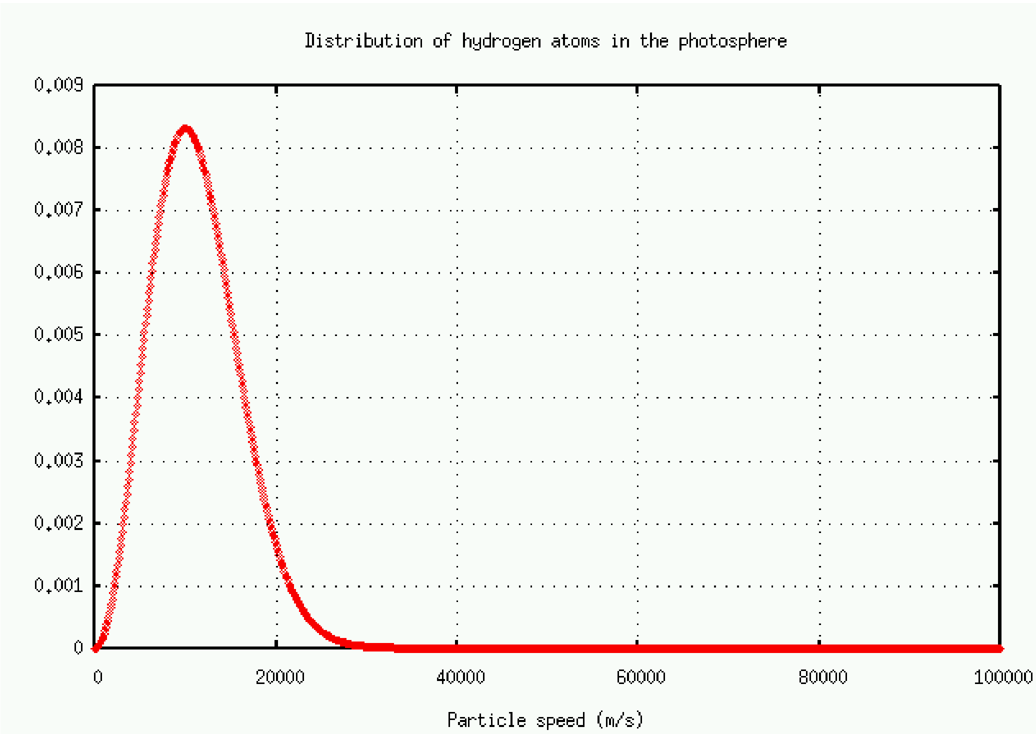


And for lines which are VERY optically thick, the equivalent width due to thermal motions grows very slowly ....



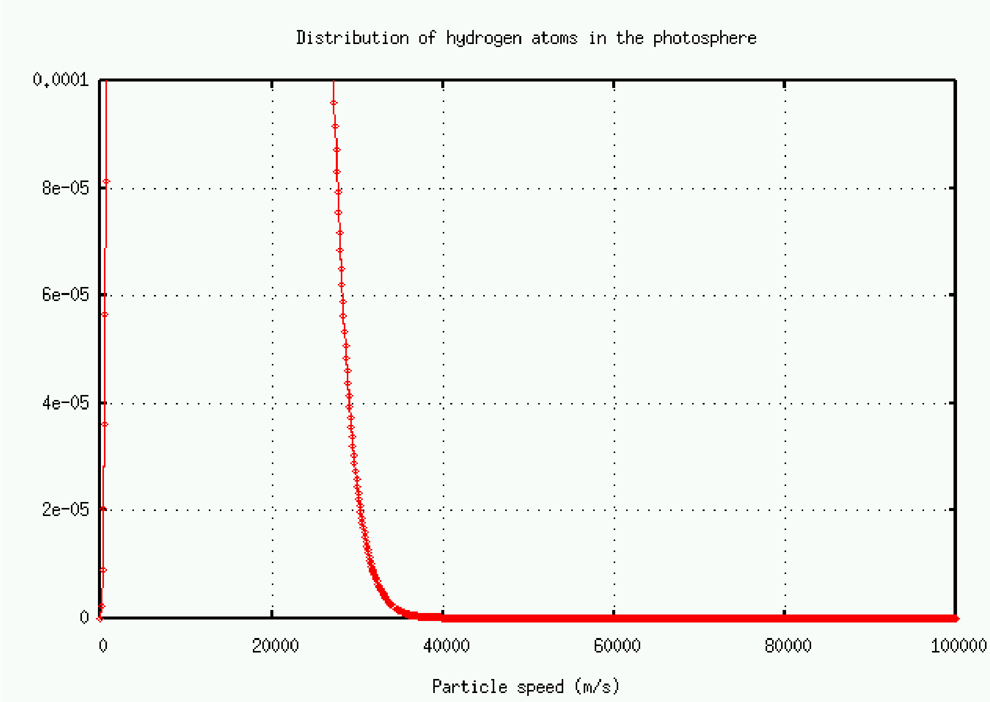
The problem here is that the only way the equivalent width can increase is for the wings to grow. The wings of this line are due to atoms which are moving either towards or away from the observer. How many atoms are moving really fast? The distributions of speeds in an ordinary cloud of gas obey a Maxwellian distribution:

$$\text{Prob}(v) = 4\pi \left(\frac{M}{2\pi RT}\right)^{3/2} v^2 e^{-Mv^2/2RT}$$

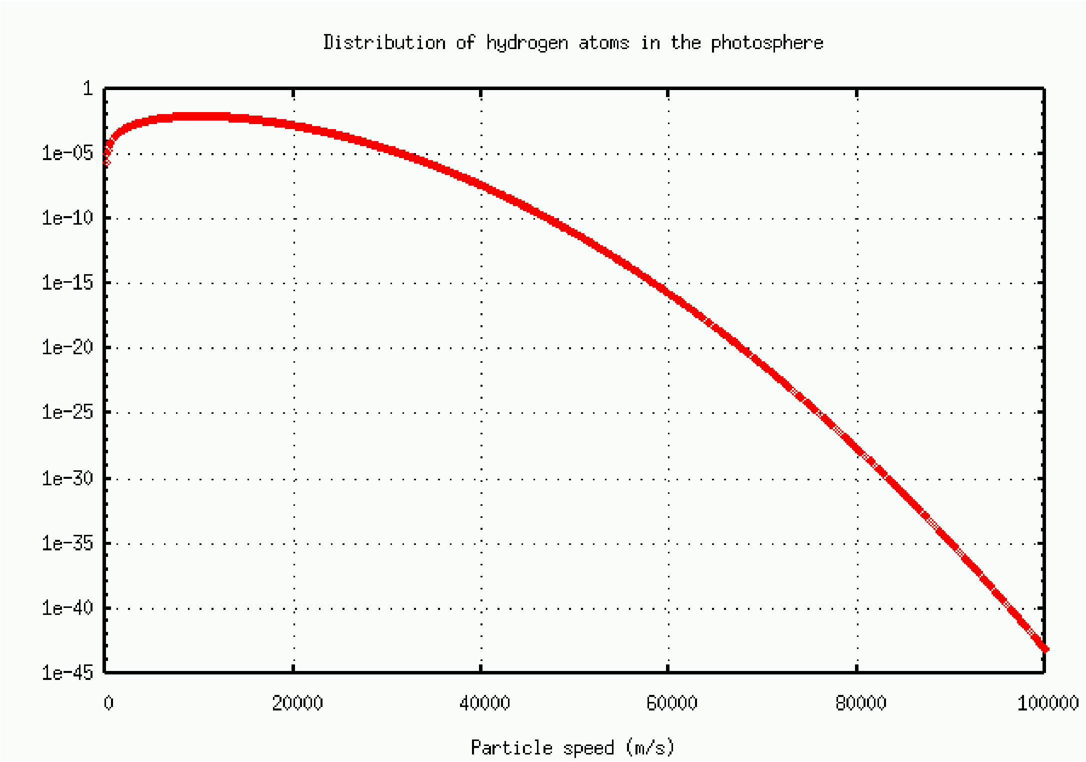


The number of atoms with speed **v** falls off at high speeds as

$$\text{Prob}(v) \propto v^2 e^{-v^2}$$



It's easier to read the numbers on a logarithmic plot:



In brief, the number of atoms with very high speeds -- and hence very large Doppler shifts -- decreases very rapidly. There just aren't many atoms available to grow the wings.

Pressure and collisionally broadened lines

The pressure and collisional (and the natural broadening) processes all produce a **Lorentzian** line profile: the optical depth $\boldsymbol{\tau}$ changes with wavelength as

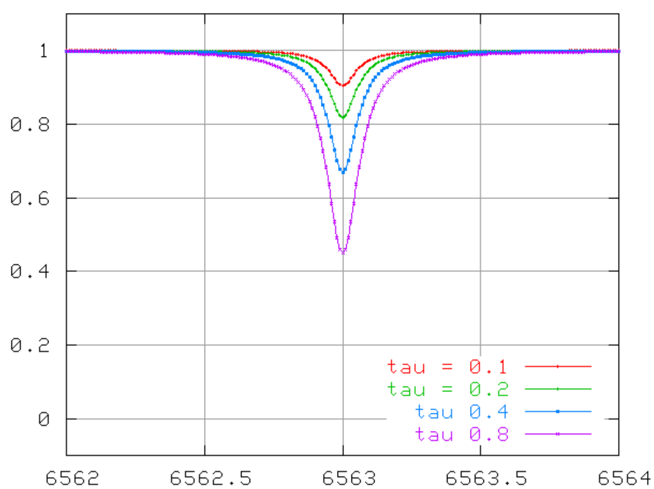
$$\tau(\lambda) = \tau(\lambda_0)\left(\frac{Q^2}{(\lambda - \lambda_0)^2 + Q^2}\right)$$

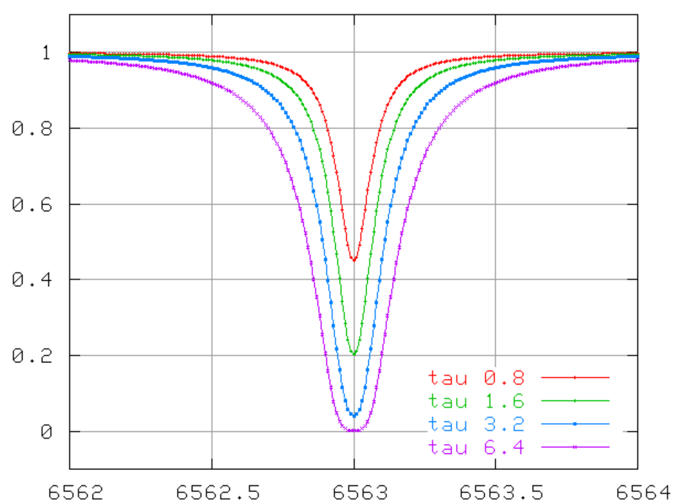
At large distances from the line center, this optical depth goes like

$$\tau \propto (\lambda - \lambda_0)^{-2}$$

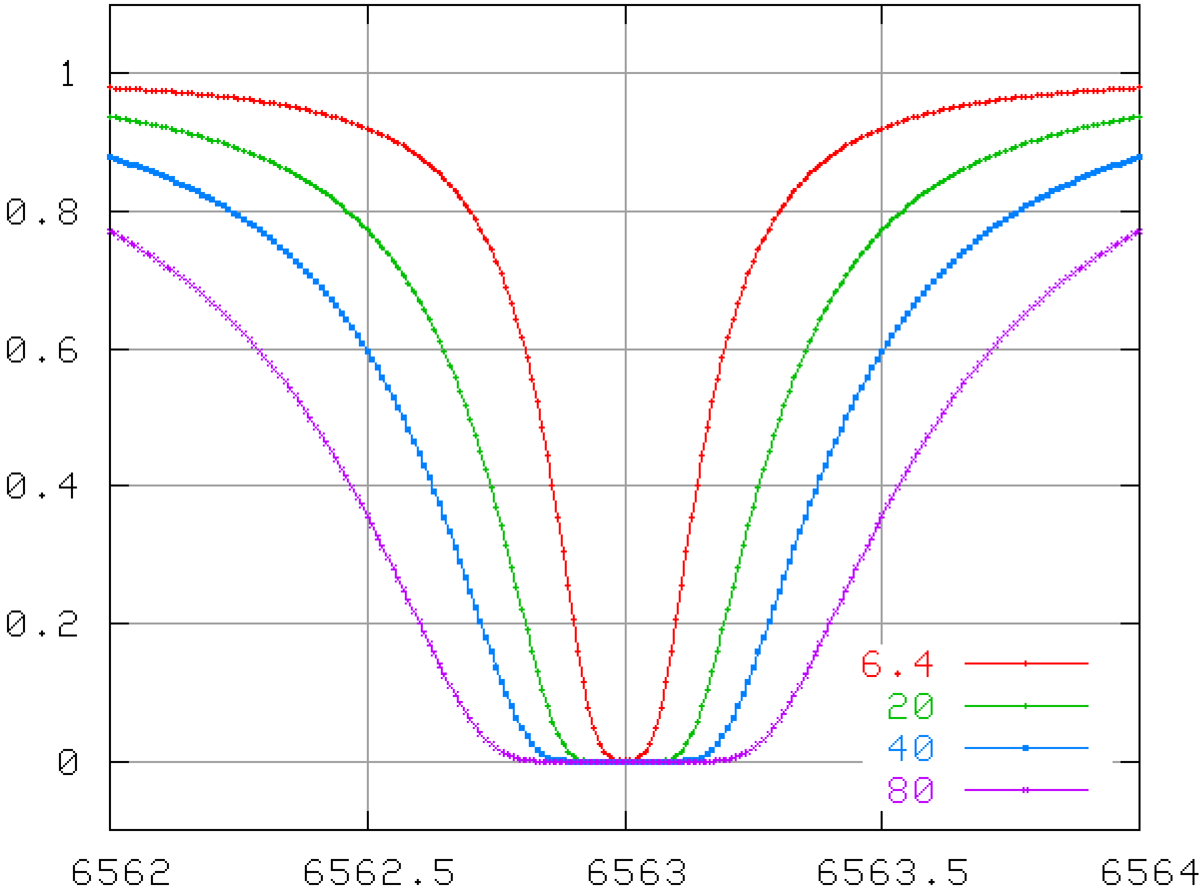
There is no exponential involved this time, so the function falls off much more slowly than that for thermal broadening. As a result, the lines have much wider wings.

Watch the lines change at small and ~ unity optical depths ...



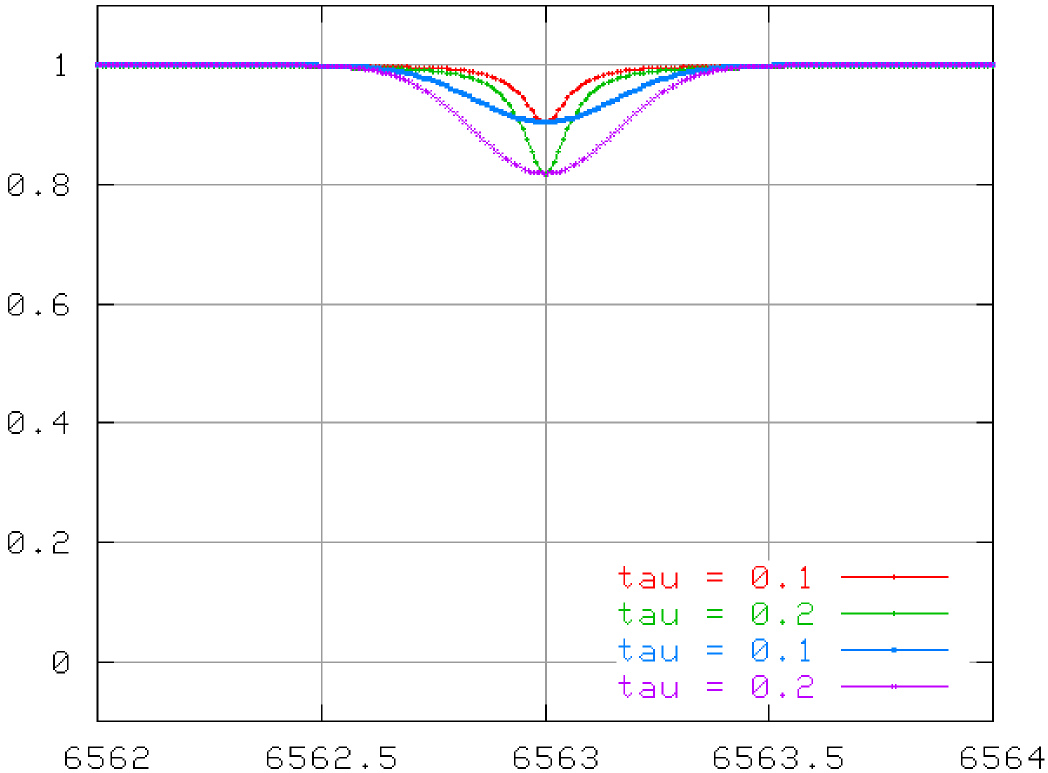


... and very large optical depths.

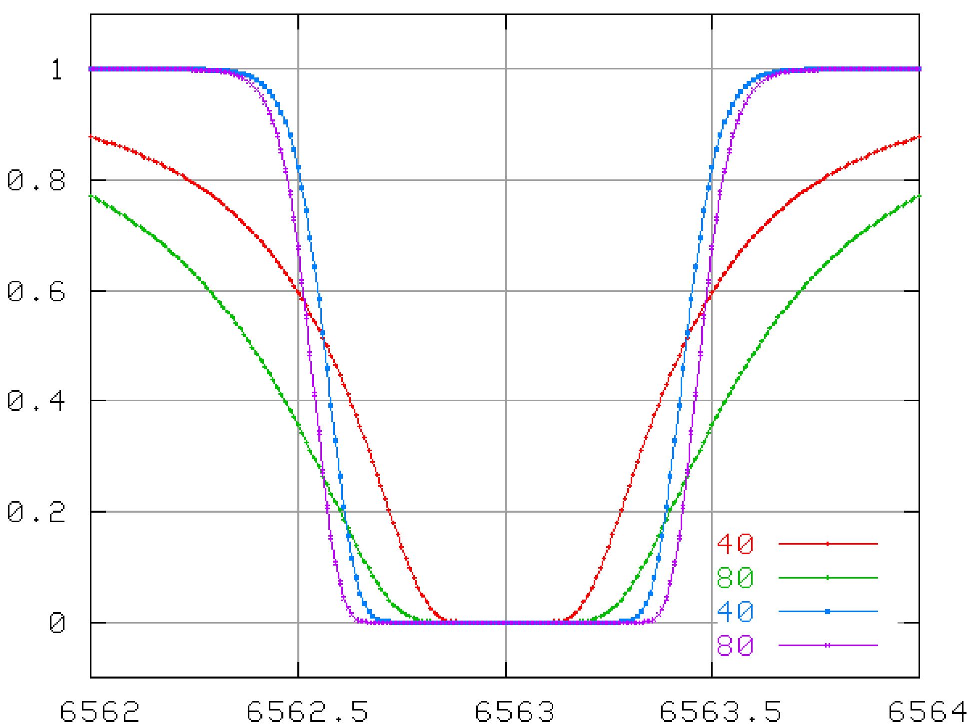


The wings continue to increase in width quickly, even after the line becomes saturated.

Compare the behavior of Doppler-broadened and collisionally-broadened lines at small optical depth



and at large optical depth.



The curve of growth

The combination of both Doppler and collisional processes leads to a complex change in the equivalent width of a spectral line as the optical depth increases. Let's use **N** to denote the number of absorbing atoms, and **W** the equivalent width of the line.

- when there are few absorbers, the optical depth is small;

$$W \propto N$$

- after the line saturates (when $\boldsymbol{\tau > 5}$ or so), the Doppler wings barely change as the number of atoms grow. To a fair approximation,

$$W \propto \sqrt{\ln N}$$

- eventually, the wings due to collisional broadening overwhelm the Doppler wings, even though the collisional term is much smaller near the line center. To a decent approximation,

$$W \propto \sqrt{N}$$

A plot of the equivalent width **W** as a function of the number of absorbing atoms **N** is called the **curve of growth**.

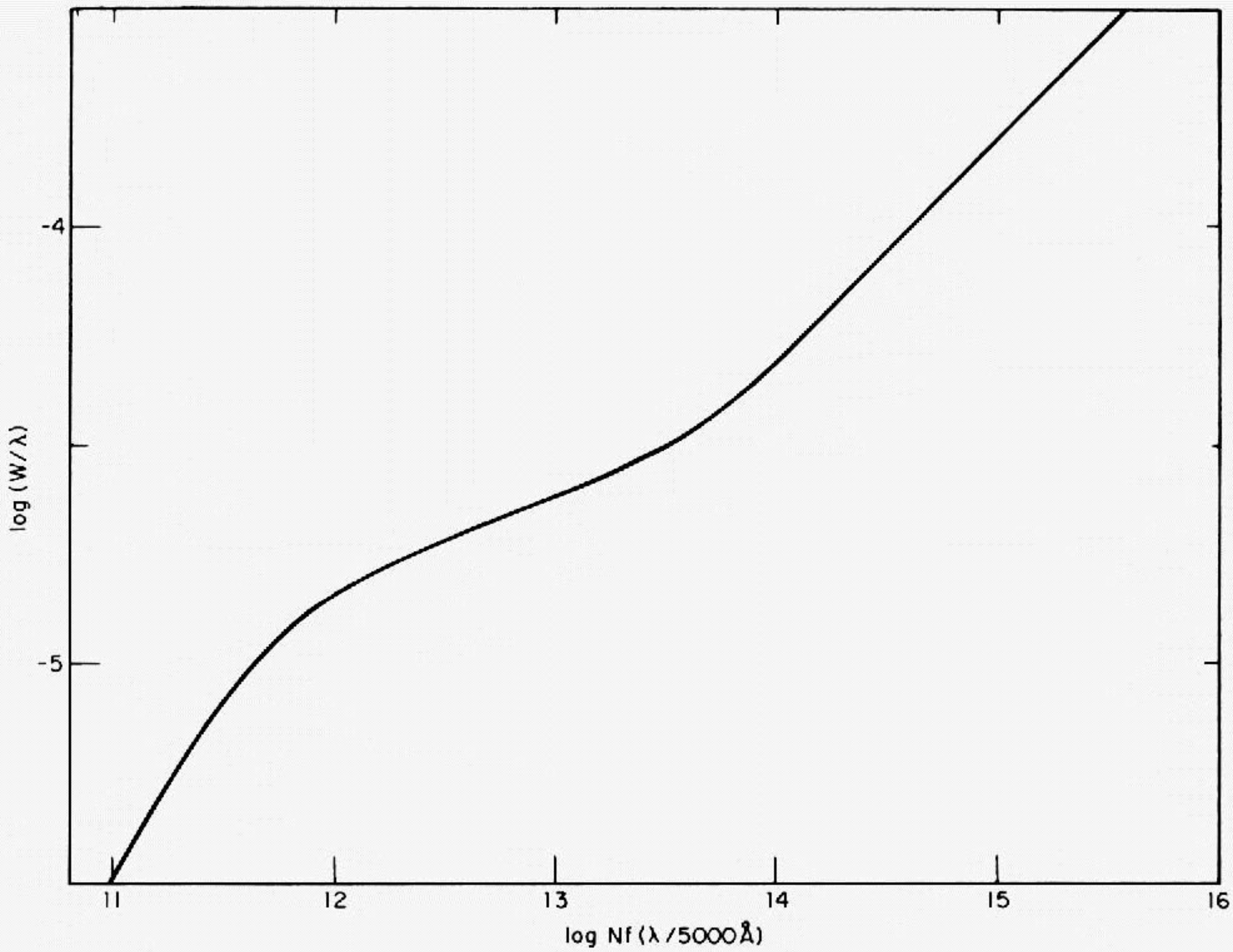


Figure 9.22 A general curve of growth for the Sun. (Figure from Aller, *Atoms, Stars, and Nebulae*, Revised Edition, Harvard University Press, Cambridge, MA, 1971.)

This figure taken from Carroll and Ostlie, "An Introduction to Modern Astrophysics" (Addison-Wesley 1996)

You can see the three different regimes: optically thin, saturated, and REALLY saturated.

Using the curve of growth

If you look closely at the figure above, you'll see that the axes are a bit peculiar.

- The vertical axis is the logarithm of (equivalent width divided by central wavelength). That means we can apply this graph to any line, not just one specific transition.
- The horizontal axis is the logarithm of (**N** times **f** times (wavelength divided by 5000 Angstroms). Let's take this apart:
  - The final piece, (wavelength divided by 5000 Angstroms), again permits us to adjust this graph to fit any absorption line.
  - The **N** refers to the **column density** of atoms which are in the proper state to absorb the photon of interest. For example, if we are studying the hydrogen Balmer alpha line, then **N** is the number of hydrogen atoms in the n=2 state. The "column density"

simply counts the number of atoms which appear in an imaginary cylindrical tube of cross-section one square centimeter running from your eye to the photosphere. We are interested in a cross-section, after all, since we are watching light from a background source pass through some column of cooler gas. What matters is not the length of the column of absorbing gas, or the density of the gas within, but the combination of those two factors.
  - The **f** refers to the **oscillator strength** of the atomic transition in question. The larger the value of **f**, the more likely the transition will occur. You can look up values of oscillator strengths for any particular transition in many references.

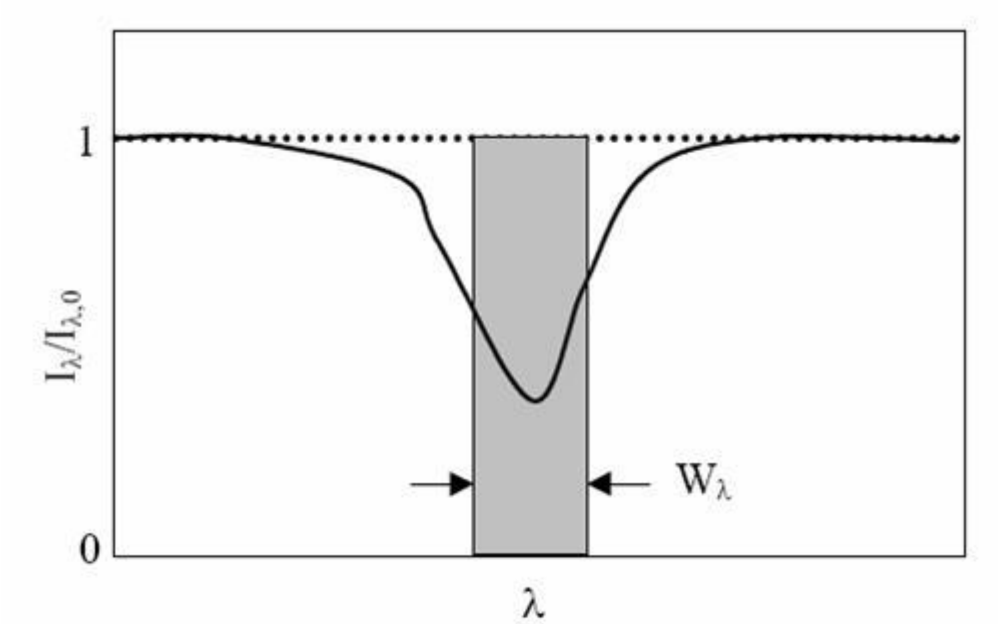
So, how do we use this? The basic idea is

1. observe some absorption line. Measure its equivalent width **W**
2. locate the position of this line on the curve of growth
3. read the horizontal value from the graph
4. look up the oscillator strength **f** for the transition in question
5. calculate the column density of atoms in the "ready-to-absorb" state
6. use the Boltzmann and Saha equations to determine the fraction of all atoms which are in the "ready-to-absorb" state
7. compute the total column density of atoms

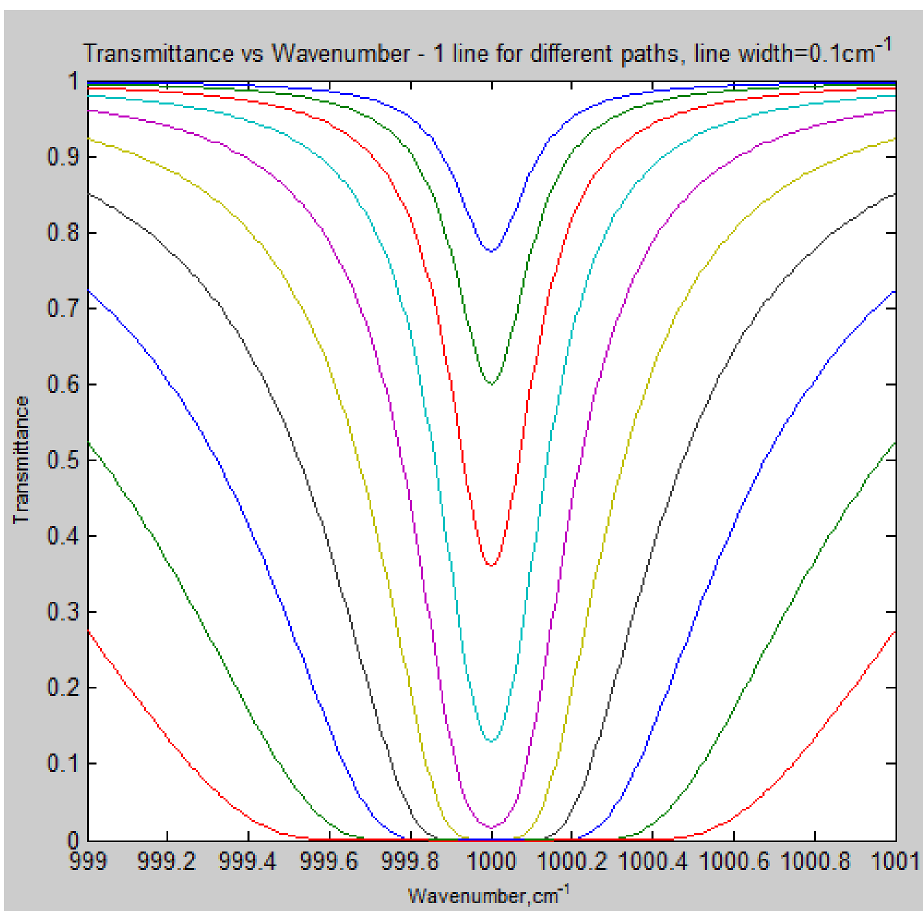
For more information

- J. B. Tatum's course on "Stellar Atmospheres" is an excellent resource, and it's on line!
- Atomic spectral line list by Hirata and Horaguchi contains a wealth of information (including oscillator strength values) for thousands of lines.

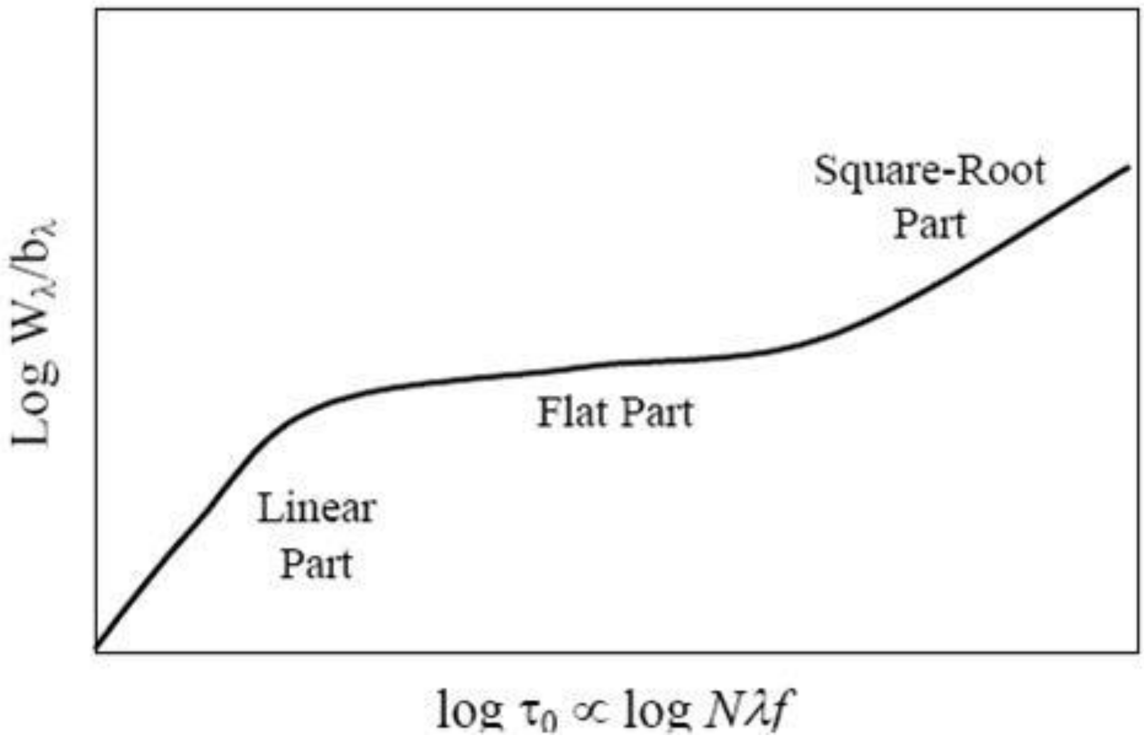
An absorption line can be characterized by its Equivalent width W.



The desire is to relate W to the lower state population (path integrated) of the absorbing species. This is problematic since absorption is bounded by I/Io = 1 (as opposed to unbounded by emission intensity).



As a result, W depends linearly on N only for small absorptions.   As the absorption increases, the variation between W and N is complex. This relation is called the Curve of Growth.



There are three general regions of the COG:

1) Linear Regime:

$$\frac{W_\lambda}{\lambda} \alpha (Nf\lambda)$$

2) Flat or Logarithmic part

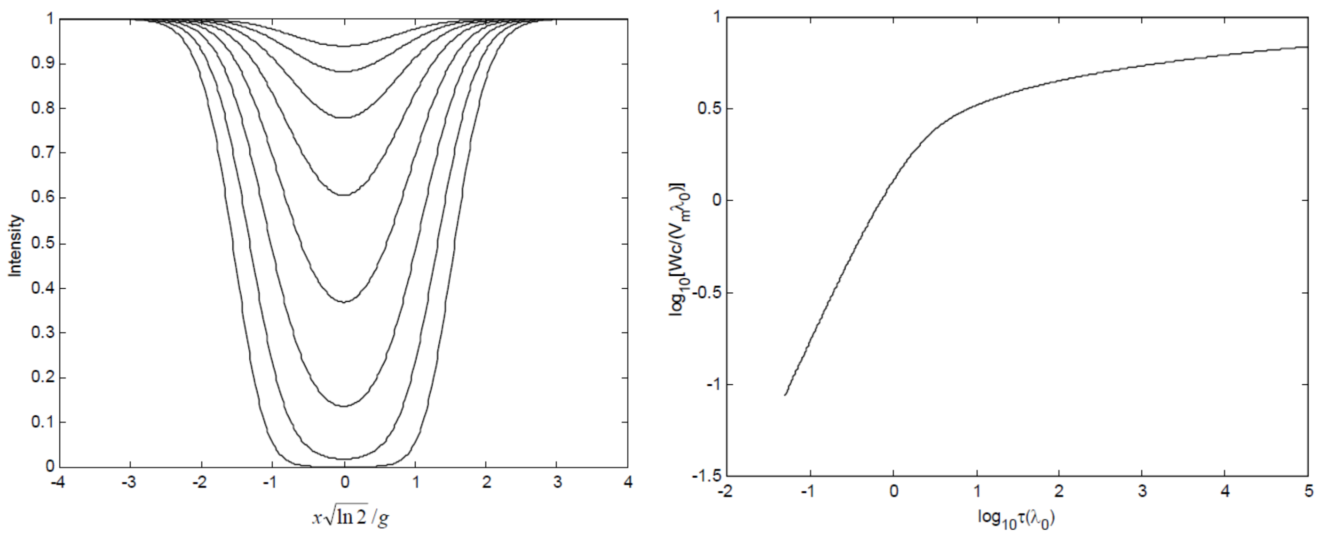
$$\frac{W_\lambda}{\lambda} \alpha \sqrt{\log(Nf\lambda)}$$

3) Square root part

$$\frac{W_\lambda}{\lambda} \alpha \sqrt{Nf\lambda}$$

The COG for any system depends on the underlying lineshape Y(l), which is determined by the broadening mechanisms (primarily Doppler and collision), and is thus a function of pressure and temperature and their distributions along the line of sight.

Guassian Profiles – stronger bend



Lorentzian Profiles – weaker bend